What is...homology intuitively?

Or: What is a hole?

The torus T and the solid torus T^s

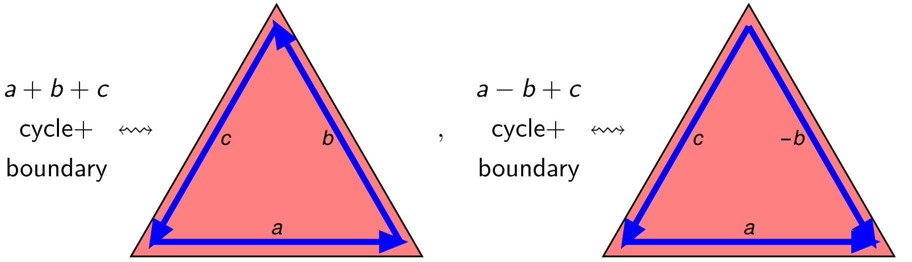
$: \begin{cases} \dim H_0 = 1 \\ \dim H_1 = 2 \\ \dim H_2 = 1 \end{cases}$

$: \begin{cases} \dim H_0 = 1 \\ \dim H_1 = 1 \\ \dim H_2 = 0 \end{cases}$

- A zero dimensional hole $\dim H_0$ is a connected component
- A one dimensional hole $\dim H_1$ is the number of necklaces you can put it on
- A two dimensional hole $\dim H_2$ is the number of plugs needed to inflate it

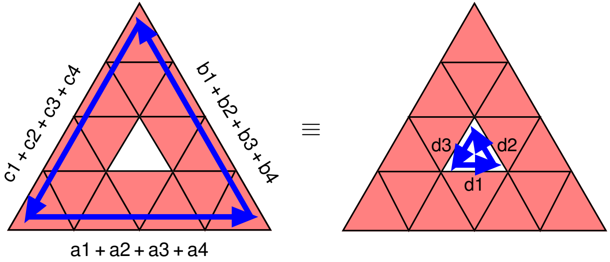
Eric Weisstein "A hole in a mathematical object is a topological structure which prevents the object from being continuously shrunk to a point."

Chains, cycles and boundaries of triangles



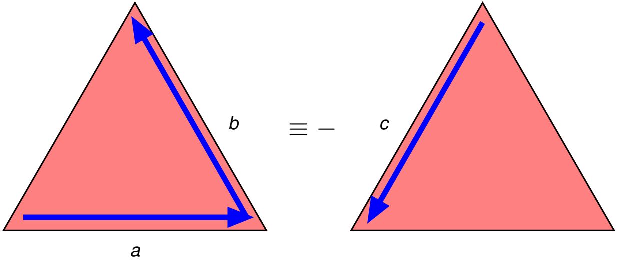
- Chains = linear combination of edges **The cells tell us what to do**
- Cycles = linear combination of edges in a triangulation going around a circle **Holes ⇒ make them potentially interesting**
- Boundary = linear combination of edges in a triangulation around a filled triangle **No holes ⇒ make them trivial**

Chains, cycles and boundaries of holed triangles



mod boundary

These are equal by successively replacing and redirecting edges in triangles:



mod boundary

For completeness: A naive but good definition

Let X be a reasonable space

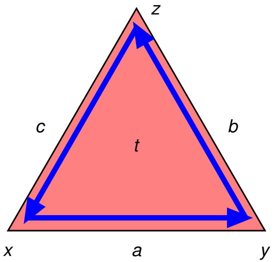
- Let $C_i(X)$ be the vector space $\mathbb{K}^n$ where n =number of i-cells Chains
- $\delta_i \colon C_i(X) \to C_{i-1}(X)$ sending an i-cell to its boundary
- Take $\ker(\delta_i)$ Cycles
- Take $\mathrm{im}(\delta_{i+1})$ Boundaries
- One checks that $\mathrm{im}(\delta_{i+1}) \subset \ker(\delta_i)$

The the ith homology $H_i(X)$ of X is the abelian group

$$H_i(X) = \ker(\delta_i)/\mathrm{im}(\delta_{i+1})$$

You can actually formulate everything using abelian groups and not vector spaces

The triangle exemplified



- $C_0 = \mathbb{Q}\{x, y, z\},\ C_1 = \mathbb{Q}\{a, b, c\},\ C_2 = \mathbb{Q}\{t\}$
- The maps are

$$\delta_3 : 0 \to C_2, 0 \mapsto 0, \quad \delta_2 : C_2 \to C_1, t \mapsto a + b + c \leftrightsquigarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\delta_1 : C_1 \to C_0, \begin{cases} a \mapsto x - y \\ b \mapsto y - z \\ c \mapsto z - x \end{cases} \leftrightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \delta_0 : C_0 \to 0, x, y, z \mapsto 0$$

- So we get $H_0(\text{triangle}) \cong \mathbb{Q},\ H_1(\text{triangle}) \cong 0,\ H_2(\text{triangle}) \cong 0$

Thank you for your attention!

I hope that was of some help.