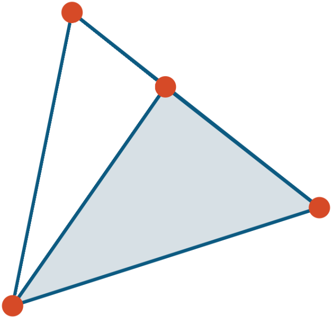
What are...simplicial and singular homology?

Or: Cycles modulo boundaries

How to count holes? – Step 1

simplicial complex $= \Delta$

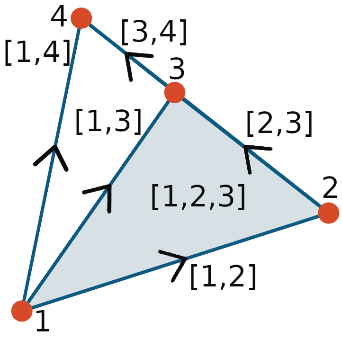


- We start by counting:

$$c_0 = \#\text{vertices} = 4 \quad c_1 = \#\text{edges} = 5 \quad c_2 = \#\text{faces} = 1$$

- Set $C_i = \mathbb{Q}^{c_i}$, with basis being vertices, edges and faces

How to count holes? – Step 2



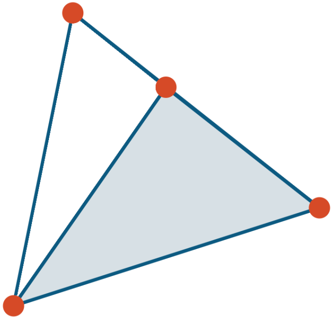
$$C_0 = \mathbb{Q}\{[0], [1], [2], [3]\}$$
$$C_1 = \mathbb{Q}\{[0,1], [0,2], [0,3], [1,2], [2,3]\}$$
$$C_2 = \mathbb{Q}\{[0,1,2]\}$$

- We construct simplices-to-faces matrices (after fixing ordered bases):

$$\delta_1 : \mathbb{Q}^5 \to \mathbb{Q}^4, \delta_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{pmatrix}, \quad \delta_2 : \mathbb{Q} \to \mathbb{Q}^5, \delta_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

- We have vector spaces $C_i = \mathbb{Q}^{c_i}$ and matrices δ_i

How to count holes? – Step 3



$H_0 = \ker(\delta_0)/\mathrm{im}(\delta_1) \cong \mathbb{Q}^1$

$H_1 = \ker(\delta_1)/\mathrm{im}(\delta_2) \cong \mathbb{Q}^1$

$H_2 = \ker(\delta_2)/\mathrm{im}(\delta_3) \cong 0$

We get a chain complex, the simplicial complex:

$$0 \xleftarrow{\;0\;} C_0 \xleftarrow{\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{pmatrix}} C_1 \xleftarrow{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}} C_2 \xleftarrow{\;0\;} 0$$

rank 0, kernel 4 | rank 3, kernel 2 | rank 1, kernel 0 | rank 0, kernel 0

Take its homology “cycles modulo boundaries”

For completeness: A formal definition

Let X be any topological space

- The nth singular chain group is

$$C_n = C_n(X) = \mathbb{Z}\{\text{singular } n\text{-simplices}\} = \mathbb{Z}\{\sigma_n \colon \Delta^n \to X\}$$

- The nth singular chain map is

$$\delta_n \colon C_n \to C_{n-1}, \quad \delta_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\dots,\underbrace{v_i}_{\text{delete}},\dots,v_n]}$$

- The ith singular homology is

$$H_n = H_n(X) = \ker(\delta_n)/\mathrm{im}(\delta_{n+1})$$

- Singular homology is a homotopy/homeomorphism invariant

Simplicial homology is what was calculated on the previous slides and

Singular homology=simplicial homology for any reasonable X

Singular homology is general, simplicial homology computable

Why the integers?

$$0 \xleftarrow{0} \underline{\mathbb{Z}} \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{0} \dots \Rightarrow H_n \cong \begin{cases} \mathbb{Z} \text{ if } n \text{ is even} \\ \mathbb{Z}/2\mathbb{Z} \text{ if } n \text{ is odd} \end{cases}$$

$$0 \xleftarrow{0} \underline{\mathbb{Q}} \xleftarrow{0} \mathbb{Q} \xleftarrow{\cdot 2} \mathbb{Q} \xleftarrow{0} \mathbb{Q} \xleftarrow{\cdot 2} \mathbb{Q} \xleftarrow{0} \dots \Rightarrow H_n \cong \begin{cases} \mathbb{Q} \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$$

$$0 \xleftarrow{0} \underline{\mathbb{Z}/2\mathbb{Z}} \xleftarrow{0} \mathbb{Z}/2\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z} \xleftarrow{0} \mathbb{Z}/2\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z} \xleftarrow{0} \dots \Rightarrow H_n \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \text{ if } n \text{ is even} \\ \mathbb{Z}/2\mathbb{Z} \text{ if } n \text{ is odd} \end{cases}$$

Underline $= C_0$

- Working over the integers one can specialize to any field **More general**
- Note that "homology=kernel-rank" does not work integrally **Careful**

Thank you for your attention!

I hope that was of some help.