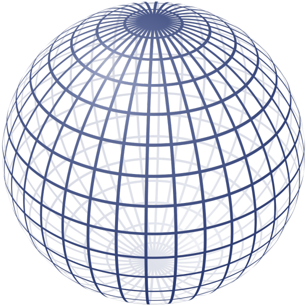
What is...a (co)homology theory?

Or: Shut up and calculate

The homology of a sphere



The singular homology of spheres is

$$H_n(S^d) \cong \begin{cases} \mathbb{Z} & n = 0, d \\ 0 & \text{else} \end{cases}$$

The proof needs only abstract properties of singular homology H_*

Brouwer's fixed point theorem – a proof sketch

Every map from the n-ball to itself has a fixed point

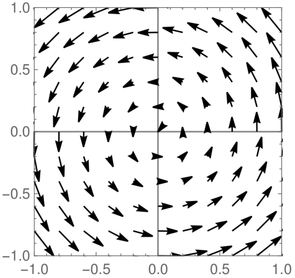
- Assume the converse and construct a map $f\colon D^n \to S^{n-1}$ with $f|_{S^{n-1}} = id$
- That can not work (for simplicity $n \geq 2$) because we would get

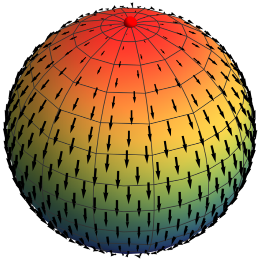
$$\mathbb{Z} \cong H_{n-1}(S^{n-1}) \xrightarrow{H_{n-1}(\iota)} H_{n-1}(D^n) \cong 0 \xrightarrow{H_{n-1}(f)} H_{n-1}(S^{n-1}) \cong \mathbb{Z}$$

with the composite $H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ being id.

The proof needs only abstract properties of singular homology H_*

The hairy ball theorem – a proof sketch





You cannot comb a hairy n-ball flat without at least one cowlick unless n is odd

- For n even we can explicitly construct vanishing vector fields as above
- A non-vanishing vector field gives a homotopy $h_t \colon S^n \to S^n$ such that

$$\Big(H_n(h_0) = 1, H_n(h_1) = (-1)^{n+1} \colon H_n(S^n) \cong \mathbb{Z} \to H_n(S^n) \cong \mathbb{Z}\Big) \Rightarrow \big(1 = (-1)^{n+1}\big)$$

The proof needs only abstract properties of singular homology H_*

For completeness: A formal definition

A homology theory H_* satisfying the dimension axiom is a functor $H_*\colon \mathrm{Top}^2 \to \mathbb{Z}\mathrm{mod}$ from pairs of topological spaces to $\mathbb{Z}$-modules together with nat. trafos $\partial = \partial_n(X, A)\colon H_n(X, A) \to H_{n-1}(A, \emptyset) = H_{n-1}(A)$ satisfying:

- Homotopic maps induce the same map in homology **Homotopy invariance**
- If (X, A) is a pair and $U \subset A$ such that its closure is contained in the interior of A, then the inclusion

$$\iota\colon (X \setminus U, A \setminus U) \to (X, A)$$

  induces an isomorphism in homology **Excision**
- Each (X, A) induces a long exact sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

  via the inclusions $i\colon A \hookrightarrow X$ and $j\colon X \hookrightarrow (X, A)$ **Exactness**
- Direct sums $\bigoplus_i H_*(X_i)$ correspond to disjoint unions $\coprod_i X_i$: they are isomorphic by the inclusions $(\iota_i)_*$ **$\oplus \leftrightsquigarrow \coprod$**
- $H_n(\text{point}) = 0$ for all $n > 0$, and $H_0(\text{point}) = \mathbb{Z}$ **Dimension axiom**

The punchline is...

- Singular homology is a homology theory satisfying the dimension axiom

  Existence

- Singular homology is up to equivalence the only such theory

  Uniqueness

- This implies that

  $$\text{singular} = \text{simplicial} = \text{cellular}$$

  for all reasonable input spaces

- One can compute *e.g.* the homology of spheres from the axioms alone and thus, proof theorems such as Brouwer's fixed point theorem and the hairy ball theorem

  No explicit arguments needed

- Cohomology can be defined dually

Dropping the dimension axiom one gets a abundance of different homology theories – all similar in flavor ("same axioms") but still different invariance

Thank you for your attention!

I hope that was of some help.