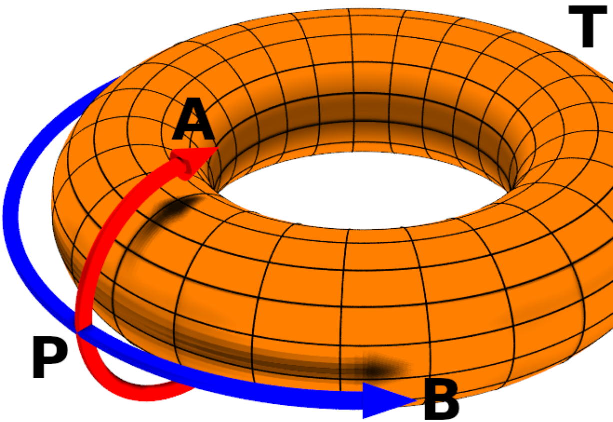
What is...the cohomology ring intuitively?

Or: Counting intersections

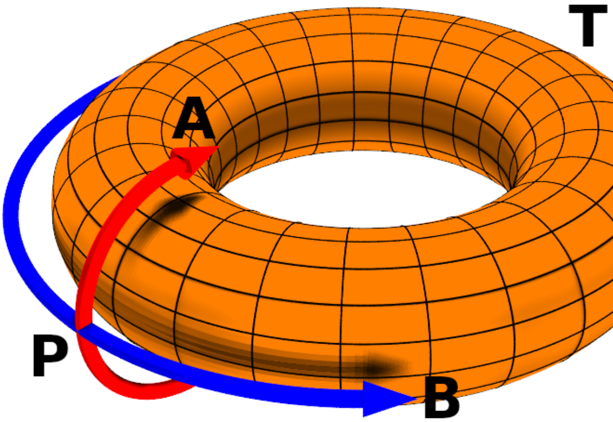
Classes and submanifolds



$$H_0(T) \cong \mathbb{Z} \leftrightsquigarrow [P], H_1(T) \cong \mathbb{Z}^2 \leftrightsquigarrow [A], [B], H_2(T) \cong \mathbb{Z} \leftrightsquigarrow [T]$$

- In good cases generators of $H_k(X)$ correspond to k-dimensional submanifolds
- We should be able to use this information to say more about X

The intersection product $\cap$



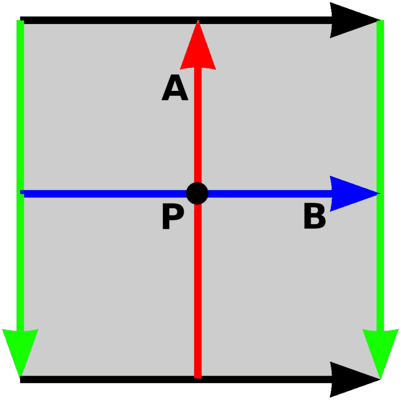
$$[A] \cap [B] = \pm[P] = [B] \cap [A]$$

Idea Submanifolds generically intersect in submanifolds $\Rightarrow$ get a product

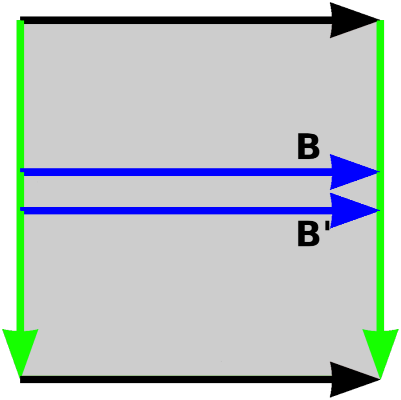
$$\cap \colon H_{n-k}(X) \times H_{n-l}(X) \to H_{n-k-l}(X)$$

($\dim X = n$) and (Codimension $k \cap$ codimension $l =$ codimension $k + l$)

A homology ring?



$[A] \cap [B] = \pm[P]$



$[B] \cap [B] = 0$

$\cap$ gives $H_*(T)$ a ring structure $H_\bullet(T)$:

$$H_\bullet(T) \xrightarrow[{[A],[B] \mapsto A,B}]{\cong} \mathbb{Z}\langle A, B\rangle / (A^2 = B^2 = 0, AB = -BA)$$

For completeness: A naive but good definition

Let X be a reasonable space, $\dim X = n$

- There is a product Intersection product

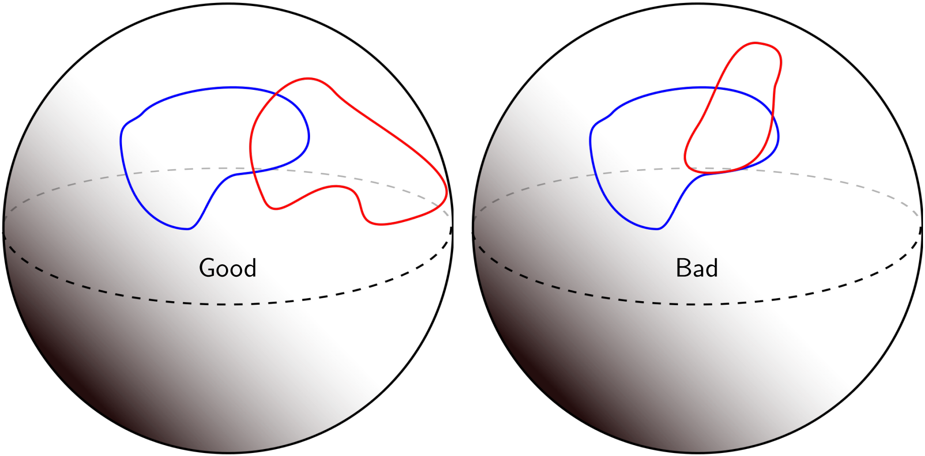
$$\cap \colon H_{n-k}(X) \times H_{n-l}(X) \to H_{n-k-l}(X), \quad [A] \cap [B] = [A \cap B]$$

- $H_\bullet = (H_*, \cap)$ is a ring "Cohomology ring"
- The ring structure is a homotopy/homeomorphism invariant

Some flaws, fixed by the "correct definition":

- This does not work for all spaces X
- $\cap$ has a non-intuitive multiplication direction
- One needs to be careful what generically intersect means Next slide

Transversality



Transverse intersection For every $p \in A \cap B$ the map of tangent bundles $T_pA \oplus T_pB \to T_pX$ induced by the inclusions is surjective

(This is only defined under appropriate smoothness conditions)

Thank you for your attention!

I hope that was of some help.