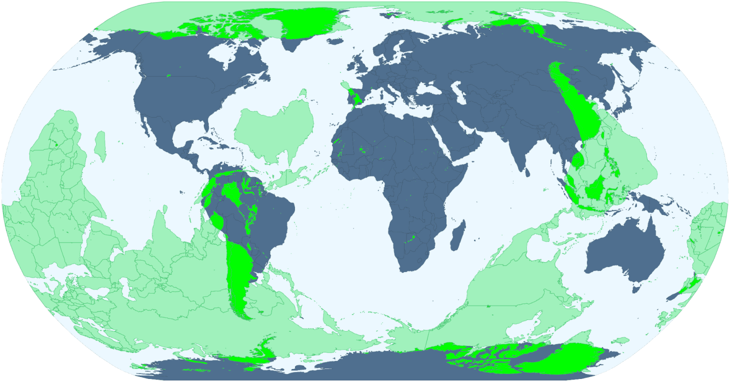
What is...the Künneth formula?

Or: Multiplication *vs.* tensor product

The real projective plane again



- $\mathbb{R}P^2$ has a cell structure with one cell in dimensions $0, 1, 2$
- The corresponding cellular chain complex $C_*(\mathbb{R}P^2)$ is

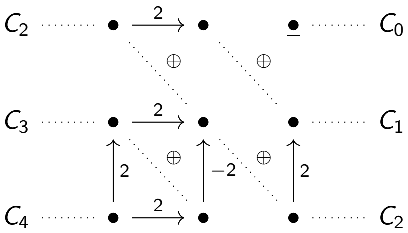
$$C_2 \cong \mathbb{Z} \xrightarrow{2} C_1 \cong \mathbb{Z} \xrightarrow{0} C_0 \cong \mathbb{Z} \quad \leftrightsquigarrow \quad \bullet \xrightarrow{2} \bullet \qquad \underline{\bullet}$$

- The corresponding cellular homology $H_*(\mathbb{R}P^2)$ is

$$\mathbb{Z} \oplus t\mathbb{Z}/2\mathbb{Z}$$

Multiplication and homology

- $\mathbb{R}P^2 \times \mathbb{R}P^2$ has a cell structure with one cell in dimensions $0, 4$, two in dimensions $1, 3$ and three in dimension 2
- The corresponding cellular chain complex $C_*(\mathbb{R}P^2 \times \mathbb{R}P^2)$ is



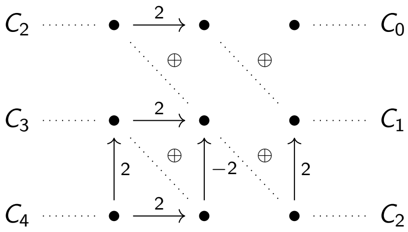
Same as $C_*(\mathbb{R}P^2) \otimes_{\mathbb{Z}} C_*(\mathbb{R}P^2)$

- The corresponding cellular homology $H_*(\mathbb{R}P^2 \times \mathbb{R}P^2)$ is

$$\mathbb{Z} \oplus t\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus t^2\mathbb{Z}/2\mathbb{Z} \oplus t^3\mathbb{Z}/2\mathbb{Z}$$

Not the same as $H_*(\mathbb{R}P^2) \otimes_{\mathbb{Z}} H_*(\mathbb{R}P^2)$

Here is the calculation



$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{\begin{pmatrix} 2 \\ 2 \end{pmatrix}} \mathbb{Z}^{\oplus 2} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 2 & -2 \\ 0 & 0 \end{pmatrix}} \mathbb{Z}^{\oplus 3} \xrightarrow{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}} \mathbb{Z}^{\oplus 2} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$

$$H_*(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong \mathbb{Z} \oplus t\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus t^2\mathbb{Z}/2\mathbb{Z} \oplus t^3\mathbb{Z}/2\mathbb{Z}$$

$$H_*(\mathbb{R}P^2) \otimes H_*(\mathbb{R}P^2) \cong \mathbb{Z} \oplus t\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus t^2\mathbb{Z}/2\mathbb{Z}$$

The Künneth measures the difference between $H_*(X) \otimes_{\mathbb{Z}} H_*(Y)$ and $H_*(X \times Y)$

For completeness: A formal statement

Let X, Y be any topological spaces, and R a PID

- There are short (non-naturally) splitting exact sequences

$$\bigoplus_{p+q=n} H_p(X,R) \otimes_R H_q(Y,R) \to H_n(X \times Y, R) \to \bigoplus_{p+q=n-1} \mathrm{Tor}^R(H_p(X,R), H_q(Y,R))$$

$$\bigoplus_{p+q=n} H^p(X,R) \otimes_R H^q(Y,R) \to H^n(X \times Y, R) \to \bigoplus_{p+q=n-1} \mathrm{Tor}^R(H^p(X,R), H^q(Y,R))$$

Note the torsion error terms

- There are isomorphism of $\mathbb{Q}$-vector spaces

$$H_*(X,\mathbb{Q}) \otimes_{\mathbb{Q}} H_*(Y,\mathbb{Q}) \cong H_*(X \times Y, \mathbb{Q})$$
$$H^*(X,\mathbb{Q}) \otimes_{\mathbb{K}} H^*(Y,\mathbb{Q}) \cong H^*(X \times Y, \mathbb{Q})$$

No error terms

- In particular, $P(X \times Y) = P(X)P(Y)$

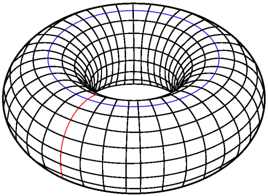
As usual: "Signs! Beware!"

If X, Y are finite cell complexes with projections $p, q \colon X \times Y \to X, Y$, then

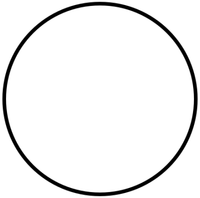
$$\times \colon H^\bullet(X, \mathbb{Q}) \otimes'_{\mathbb{Q}} H^\bullet(Y, \mathbb{Q}) \xrightarrow{\cong} H^\bullet(X \times Y, \mathbb{Q})$$

as graded commutative rings

$$x(a, b) = p_*(a) \smile q_*(b)$$



$$H^\bullet(T^d, \mathbb{Q}) \cong \bigwedge \mathbb{Q}^d \quad \deg X_i = 1$$



$$H^\bullet(S^1, \mathbb{Q}) \cong \mathbb{Q}[X]/(X^2) \quad \deg X = 1$$

Thank you for your attention!

I hope that was of some help.