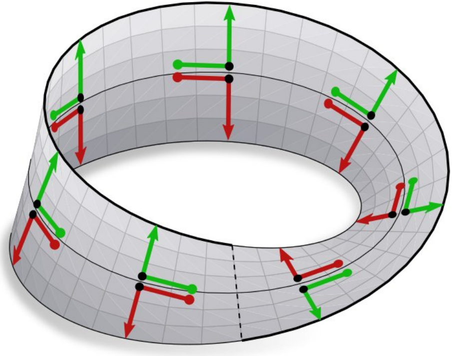
What is...orientability?

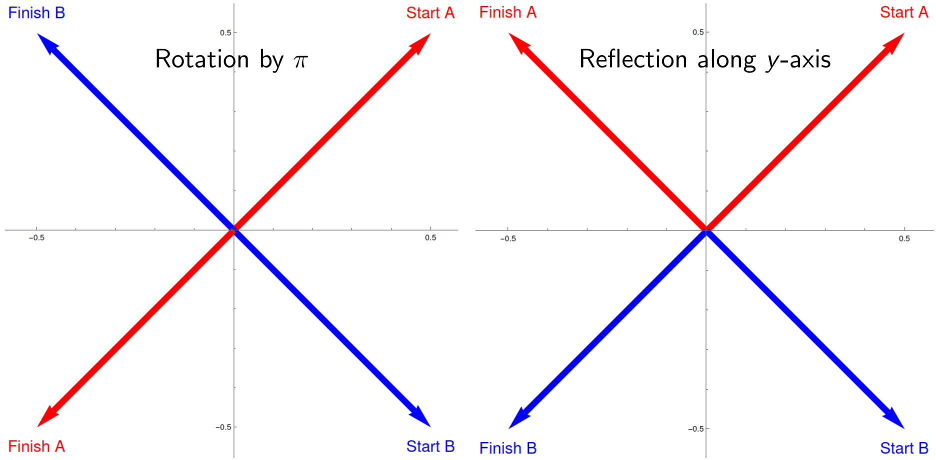
Or: A homological definition

The Möbius strip again



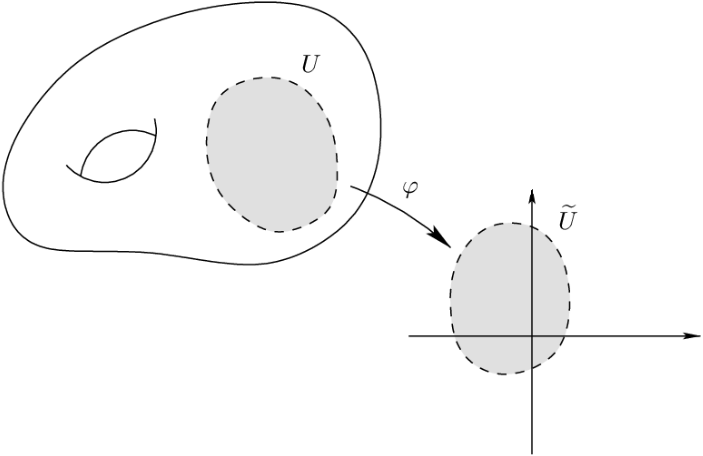
- Orientability of a manifold is a consistent choice of a coordinate system per point
- There are non-orientable manifolds
- What can homology say about orientability?

My wish list for orientation



- An orientation should be preserved under rotation and translation and scaling
- An orientation should be reversed under reflection

Homology detects $\mathbb{R}^n$ locally



- By local triviality of an n-manifold M one gets

$$H_n(M, M \setminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$$

- Rotations/reflections give maps from $H_{n-1}(S^{n-1})$ to itself, satisfying

$\text{Rotation}_*(\pm 1) = \pm 1$ $\text{Reflection}_*(\pm 1) = \mp 1$

For completeness: A formal definition

Let M be an n-manifold

- A local orientation at $x \in M$ is a choice $\alpha_x = \pm 1 \in H_n(M, M \setminus \{x\})$
- A (global) orientation is a consistent choice of α_x for all x, meaning:

$$\forall x \in M \; \exists \text{ open } U \cong D^n \subset \mathbb{R}^n \text{ containing } x \text{ such that}$$
$$\exists \alpha_U = \pm 1 \in H_n(M, M \setminus U) \cong \mathbb{Z} \text{ with}$$
$$\forall y \in U\colon (\iota_y)_*\colon H_n(M, M \setminus U) \to H_n(M, M \setminus \{y\}), \quad \alpha_U \mapsto \alpha_y$$

  where $\iota_y\colon (M, M \setminus U) \to (M, M \setminus \{y\})$ is the inclusion

- If an orientation exists for M, then M is called orientable

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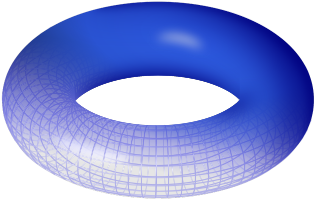
- The second point should be read as "Every x has a neighborhood in which the orientation is rotated or is translated or scaled but not reflected

  Compatibility condition formulated homologically
- The same definition works for homology with coefficients in an arbitrary PID

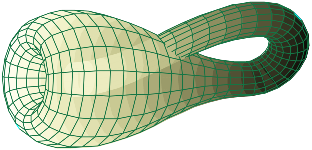
This works well

Theorem If M is a closed connected n-manifold, then

$$H_n(M) \cong \begin{cases} \mathbb{Z} \text{ if } M \text{ is orientable} \\ 0 \text{ if } M \text{ is not orientable} \end{cases}$$



$$H_*(\text{torus}) \cong \mathbb{Z} \oplus t\mathbb{Z}^{\oplus 2} \oplus t^2\mathbb{Z}$$



$$H_*(\text{Klein bottle}) \cong \mathbb{Z} \oplus t(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus t^2 0$$

Thank you for your attention!

I hope that was of some help.