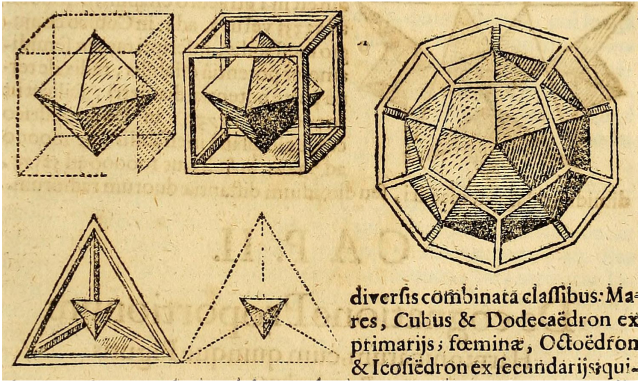
What is...Poincaré duality?

Or: My face is 0-dimensional

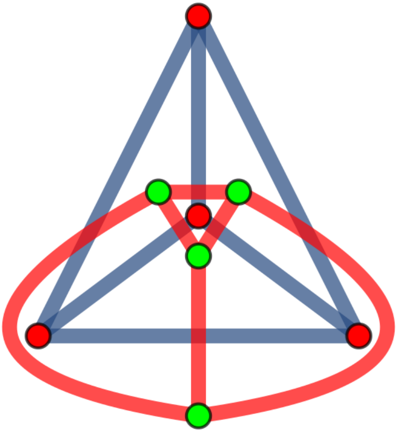
Kepler's Harmonices Mundi ∼1619



- The dual of a tetrahedron is a tetrahedron
- The dual of a cube is a octahedron
- The dual of a dodecahedron is a icosahedron

What does this mean?

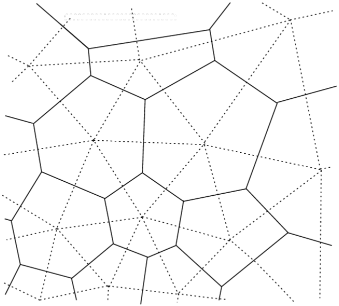
Dual graph



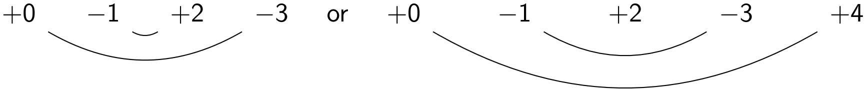
The dual G^* of a plane graph G_* is obtained by reversing dimensions:

- G^* has a vertex for each face of G_*
- G^* has an edge for each edge of G_*; connecting adjacent faces
- G^* has a face for each vertex of G_*

Dual Euler characteristic



- Similarly, for any cell complex X_* one can define a dual cell complex X^*
- We have $\chi(X_*) = \pm\chi(X^*)$ since



What happens on (co)homology?

For completeness: A formal statement

If M is an orientable closed n-manifold, then for $0 \leq k \leq n$:

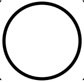
$$[M] \frown \_ : H^k(M) \xrightarrow{\cong} H_{n-k}(M)$$

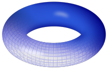
- Here $\frown$ is the pairing

$$\_ \frown \_ : H_k(M) \times H^l(M) \to H_{k-l}(M), \sigma \frown \phi = \phi(\sigma|[v_0, ..., v_l])\sigma|[v_l, ..., v_k]$$

- There are many generalization, *e.g.* relaxing “orientable” or “closed”

This implies that the Hilbert–Poincaré polynomial of M is palindromic:

P () $= 1 + t \leftrightsquigarrow$ 1 t

P () $= 1 + 2t + t^2 \leftrightsquigarrow$ 1 $2t$ t^2

$P(\mathbb{C}P^6) = 1 + t^2 + t^4 + t^6 \leftrightsquigarrow$ 1 0 t^2 0 t^4 0 t^6

Wait! How do you see the palindromic property?

The universal coefficient theorem (UCT) for cohomology for all X and PID R:

$$0 \to \mathrm{Ext}\big(H_{k-1}(X), R\big) \to H^k(X, R) \to \hom\big(H_k(X), R\big) \to 0$$

is a split (non-naturally) short exact sequence

- Thus, in general

$$H^k(X) \cong \hom\big(H_k(X), \mathbb{Z}\big) \oplus \mathrm{Ext}\big(H_{k-1}(X), \mathbb{Z}\big)$$

- Ext vanishes over $\mathbb{Q}$ and $\hom\big(H_k(X), \mathbb{Q}\big) \cong H_k(X, \mathbb{Q})$ if finite, which implies

$$H_k(M, \mathbb{Q}) \cong H^k(M, \mathbb{Q})$$

- Paste this together with Poincaré duality:

$$H_k(M, \mathbb{Q}) \cong H^k(M, \mathbb{Q}) \cong H_{n-k}(M, \mathbb{Q})$$

Thank you for your attention!

I hope that was of some help.