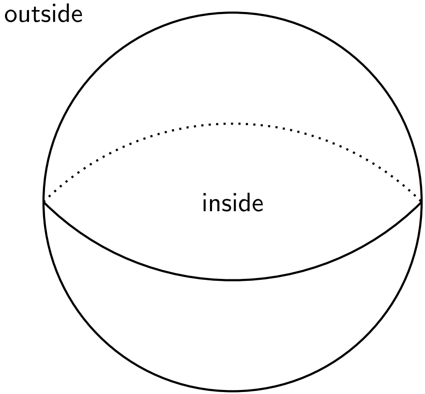
What is...Alexander duality?

Or: Horned spheres!?

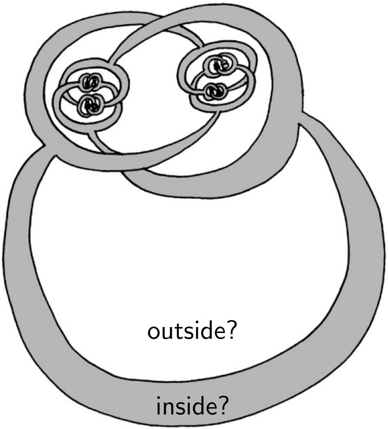
A harmless sounding statement...



A sphere S^2 embedded in $\mathbb{R}^3$ divides $\mathbb{R}^3$ into an inside and an outside

Formally $\mathbb{R}^3 \setminus \iota(S^2)$ is has two connected components for any $\iota\colon S^2 \hookrightarrow \mathbb{R}^3$

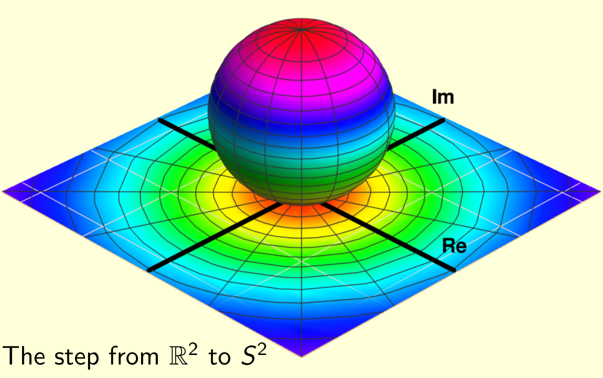
...is not harmless at all!



A sphere S^2 embedded in $\mathbb{R}^3$ divides $\mathbb{R}^3$ into an inside and an outside. Really?

The more one thinks about it, the less clear it becomes!

A homological formulation



The step from $\mathbb{R}^2$ to S^2

- We can replace $\mathbb{R}^3$ with S^3 Stereographic Projection
- The number of connected component of $S^3 \setminus \iota(S^2)$ is $\dim H_0\big(S^3 \setminus \iota(S^2)\big)$
- Hence, reduced homology should satisfy

$$\dim \tilde{H}_0\big(S^3 \setminus \iota(S^2), \mathbb{Q}\big) = 1$$

- So we need to compute $\dim \tilde{H}_0\big(S^3 \setminus \iota(S^2), \mathbb{Q}\big)$

For completeness: A formal statement

If $\emptyset \subsetneq K \subsetneq S^n$ is a compact and locally contractible, then

$$\tilde{H}_i(S^n \setminus K) \xrightarrow{\cong} \tilde{H}^{n-i-1}(K)$$

- This only depends on intrinsic properties of K
- For $K = \iota(S^{n-1}) \cong S^{n-1}$ one gets

$$\tilde{H}_0(S^n \setminus S^{n-1}) \xrightarrow{\cong} \tilde{H}^{n-1}(S^{n-1}) \cong \mathbb{Z}$$

- Thus, we get

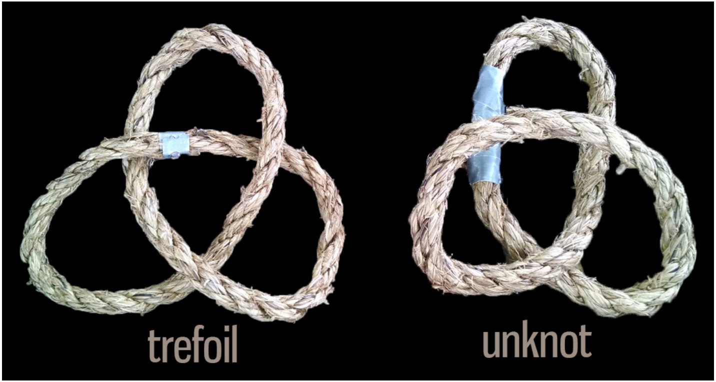
$$\dim \tilde{H}_0(S^n \setminus \iota(S^{n-1}), \mathbb{Q}) = 1$$

- This is a consequence of (the a bit more general)

$$H_i(M, M \setminus K) \xrightarrow{\cong} H^{n-i}(K)$$

where M is closed orientable n-manifold and where $K \subset M$ is compact and locally contractible

Knots? Not quite...



- A knot K is an embedding $S^1 \hookrightarrow S^3 \rightsquigarrow$ thickened into a torus $\overline{K} \cong T$ A rope
- One gets

$$\tilde{H}_i(S^n \setminus \overline{K}) \xrightarrow{\cong} \tilde{H}^{n-i-1}(\overline{K}) \cong \tilde{H}^{n-i-1}(T)$$

- This does not depend on the embedding, so can not distinguish knots

Thank you for your attention!

I hope that was of some help.