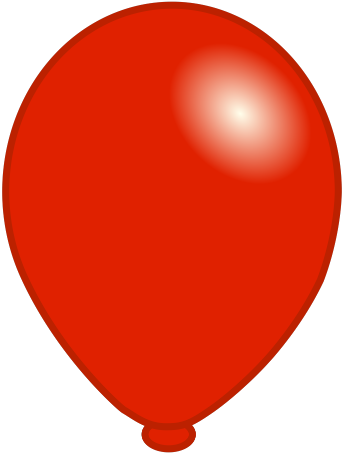
What are...examples of (co)homology groups?

Or: Another of my favorite lists

We start with everyone's(?) favorite space

$H^\bullet\Big(S^n =$

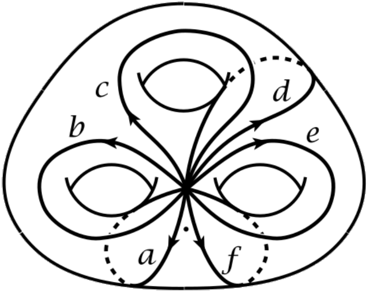


$\Big) \cong \mathbb{Z}[X]/(X^2), \deg X = n$

- This can be computed directly from the cell structure Balloon
- For n odd $\mathbb{Z}[X]/(X^2)$ isn't quite right to write as a graded commutative ring
  This is mostly ignored in this video

Surfaces of genus g

$H^\bullet \Big(M_{g,0} =$



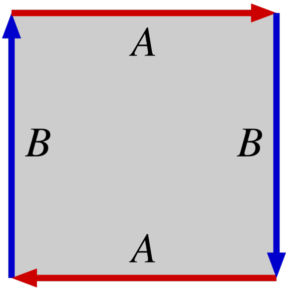
$\Big) \cong \dfrac{\mathbb{Z}[X_1, \ldots, X_{2g}]}{(X_i X_j = -X_j X_i = (1 - \delta_{i,j}) X_1 X_2)}$

$\deg X_i = 1$

- This can be computed intersecting submanifolds Intersection ring
- X_i correspond to the classes $[\alpha_i]$ of the fundamental loops
- $X_1 X_2$ corresponds to the class $[M_{g,0}]$ of the beast itself

Real projective spaces

$$H^\bullet\Big(\mathbb{R}P^n =$$



$$, \mathbb{Z}/2\mathbb{Z}\Big) \cong \frac{\mathbb{Z}/2\mathbb{Z}[X]}{(X^{n+1})}, \deg X = 1$$

- This can be computed intersecting submanifolds Intersection ring
- $n = 5$ $[\mathbb{R}P^1] \leftrightsquigarrow X, \ldots, [\mathbb{R}P^5] \leftrightsquigarrow X^5$
- Even nicer $H^\bullet(\mathbb{C}P^n) \cong \mathbb{Z}[X]/(X^{n+1})$, $\deg X = 2$

For completeness: A list

Here is a list of important cohomology rings

- Spheres S^n

$$H^\bullet(S^n) \cong \frac{\mathbb{Z}[X]}{(X^2)}, \deg X = n$$

- Torus T, real projective plane $\mathbb{R}P^2$ and Klein bottle K ($\deg X = \deg Y = 1$)

$$H^\bullet(T) \cong \bigwedge\{X, Y\},\ H^\bullet(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z}) \cong \frac{\mathbb{Z}/2\mathbb{Z}[X]}{(X^3)},\ H^\bullet(K, \mathbb{Z}/2\mathbb{Z}) \cong \frac{\mathbb{Z}/2\mathbb{Z}[X, Y]}{(X^3, Y^2, X^2Y)}$$

- Orientable surfaces $M_{g,0}$ of genus $g > 0$ without boundary

$$H^\bullet(M_{g,0}) \cong \frac{\mathbb{Z}[X_1, \dots, X_{2g}]}{(X_iX_j = -X_jX_i = (1 - \delta_{i,j})X_1X_2)}, \deg X_i = 1$$

- Real and complex projective spaces

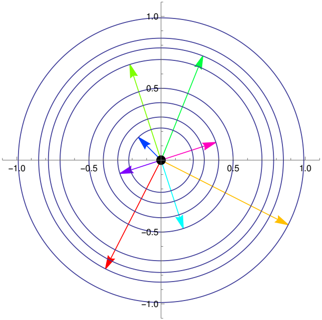
$$H^\bullet(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) \cong \frac{\mathbb{Z}/2\mathbb{Z}[X]}{(X^{n+1})}, \deg X = 1, \quad H^\bullet(\mathbb{C}P^n) \cong \frac{\mathbb{Z}[X]}{(X^{n+1})}, \deg X = 2$$

- Various topological groups $G/\mathbb{C}$ ($\deg X_i = i$)

G	$\mathrm{U}(n)$	$\mathrm{SU}(n)$	$\mathrm{Sp}(n)$	$\mathrm{SO}(n)$
$H^\bullet$	$\bigwedge\{X_1, X_3, \dots, X_{2n-1}\}$	$\bigwedge\{X_3, X_5, \dots, X_{2n-1}\}$	$\bigwedge\{X_3, X_7, \dots, X_{4n-1}\}$	next slide

Special orthogonal groups

$H^\bullet\Big(\mathrm{SO}_n(\mathbb{R}) =$



$, \mathbb{Z}/2\mathbb{Z}\Big) \cong \bigotimes_{i \text{ odd}} \frac{\mathbb{Z}/2\mathbb{Z}[X_i]}{(X_i^{2^{k_i}})}$

k_i min with $i2^{k_i} \geq n$

$\deg X_i = i$

- Use a nice cellular map $\prod_i \mathbb{R}P^i \to \mathrm{SO}_n(\mathbb{R})$ "Rotation equals rotation axis"
- $n = 2$ $H^\bullet(\mathrm{SO}_2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \cong H^\bullet(S^1, \mathbb{Z}/2\mathbb{Z}) \cong \frac{\mathbb{Z}/2\mathbb{Z}[X]}{(X^2)}$, $\deg X = 1$
- $n = 3$ $H^\bullet(\mathrm{SO}_3(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \cong H^\bullet(\mathbb{R}P^3, \mathbb{Z}/2\mathbb{Z}) \cong \frac{\mathbb{Z}/2\mathbb{Z}[X]}{(X^4)}$, $\deg X = 1$

Thank you for your attention!

I hope that was of some help.