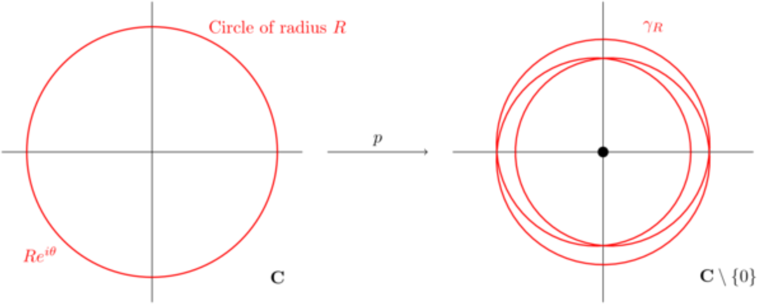
What are...some applications of topology?

Or: Applications 1 (topology in mathematics).

The fundamental theorem of algebra

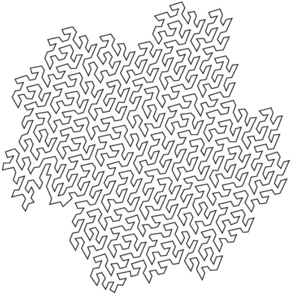


Proof via fundamental group π_1!

- Assume $f = z^n + a_{n-1}z^{n-1} + ... + a_0, a_i \in \mathbb{C}$ with $n > 0$ has no roots
- Use f to define a function p on $\mathbb{C} \setminus \{0\}$
- p corresponds to $[z^n]$ and at the same time $[1]$ in $\pi_1(\mathbb{C} \setminus \{0\})$
- $\pi_1(\mathbb{C} \setminus \{0\}) \cong \pi_1(S^1) \cong \mathbb{Z}$ and $[z^n] \leftrightsquigarrow n$ while $[1] \leftrightsquigarrow 0$ so $n = 0$

Jordan–Brouwer separation theorem

$S^1 \hookrightarrow \mathbb{R}^2$:



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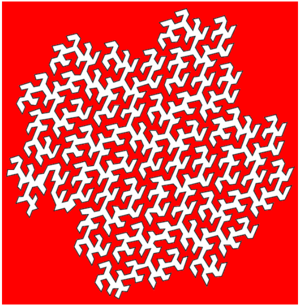


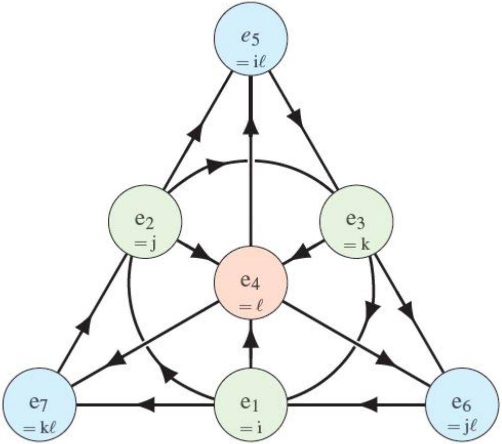
Proof via (co)homology H_*, H^* and Alexander duality!

- Every embedding of $S^{n-1} \hookrightarrow \mathbb{R}^n$ divides $\mathbb{R}^n$ in interior and exterior
- This theorem is far from being obvious, *e.g.* mind space-filling curves
- The proof via Alexander duality (replacing $\mathbb{R}^n$ by S^n)

$$\Big(\tilde{H}_0(S^n \setminus \iota(S^{n-1})) \xrightarrow{\cong} \tilde{H}^{n-1}(S^{n-1}) \cong \mathbb{Z}\Big) \Rightarrow \Big(\dim \tilde{H}_0(S^n \setminus \iota(S^{n-1}), \mathbb{Q}) = 1\Big)$$

is general and "straightforward"

The only division algebras over $\mathbb{R}$ are $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

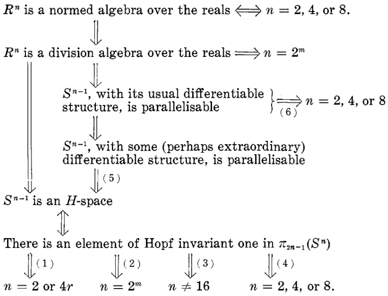


Proof via the cohomology ring $H^\bullet$!

- $\mathbb{R}^n$ is a normed algebra only for $n = 1, 2, 4, 8$
- The proof uses the Hopf invariant of maps $f\colon S^m \to S^n$ obtained from $H^\bullet$
- This is a topological proof of a purely algebraic statement

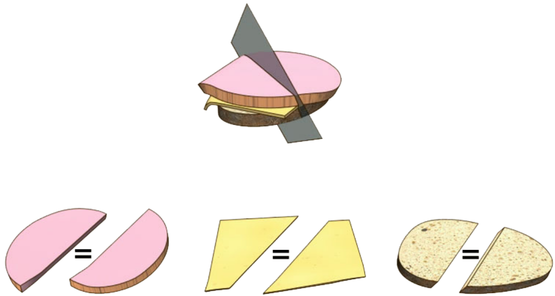
For completeness: A formal statement

There is a whole zoo of very similar statements:



- The case $n = 1$ is exceptional (but easy) and usually excluded
- Some of the statements above are algebraic some topological in nature
- Beware "The only division algebras over $\mathbb{R}$ are $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$" does not quite follow and needs some extra work/reformulation

Honorable mentions



- Brouwer fixed point + hairy ball Can be proven using H_*
- Infinitude of primes Can be proven using basic topology
- Nielsen–Schreier Can be proven using covering theory
- Cayley–Hamilton Can be proven using basic topology
- Borsuk–Ulam + ham sandwich Can be proven using H_*
- Many more...

Thank you for your attention!

I hope that was of some help.