What are...covering space action?

Or: Deck transformations and friends

Groups in the wild

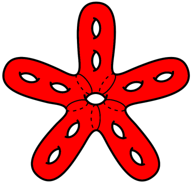
Groups naturally arise as automorphisms a.k.a. symmetries of objects, *e.g.*:

- Symmetry groups of the platonic solids Dice!

$S_4 \circlearrowright$  $S_4 \times \mathbb{Z}/2\mathbb{Z} \circlearrowright$ 

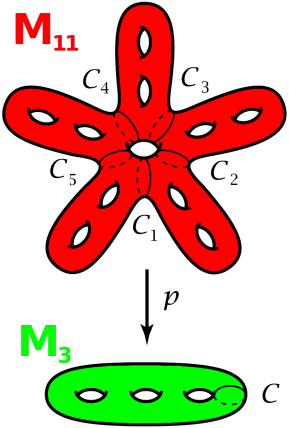
$A_5 \times \mathbb{Z}/2\mathbb{Z} \circlearrowright$ 

- Topological spaces often have symmetries, *e.g.*

$\mathbb{Z}/5\mathbb{Z} \circlearrowright$ 

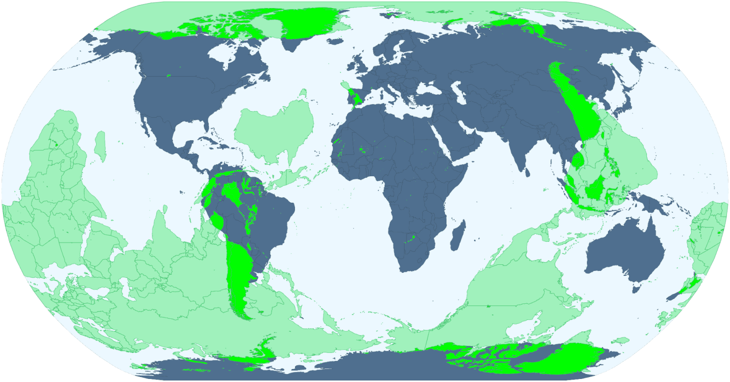
Coverings are crucially related to groups, so:

Question What is a relation between coverings and group actions?



- The surface M_{11} of genus 11 has a $G = \mathbb{Z}/5\mathbb{Z}$ symmetry Groups action
- Identifying along orbits gives $M_{11}/G \simeq M_3$ the surface of genus 3 Quotient
- M_{11} has a projection map to $M_{11}/G \simeq M_3$ Covering

Antipodes and coverings



- S^2 has a $G = \mathbb{Z}/2\mathbb{Z}$ symmetry given by $x \mapsto -x$ Groups action
- Identifying along orbits gives $S^2/G \simeq \mathbb{R}P^2$ the real projective plane Quotient
- S^2 has a projection map to $S^2/G \simeq \mathbb{R}P^2$ Covering

For completeness: A formal definition/statement

An action of a group G on a topological space X is a homomorphism

$$G \to \mathrm{Homeo}(X) = \{f : X \to X \mid f \text{ homeomorphism}\}$$

$\mathrm{Homeo}(X)$ equals topological symmetries of X

Form a topological space X/G whose points are orbits $\{g.x \mid g \in G\}$

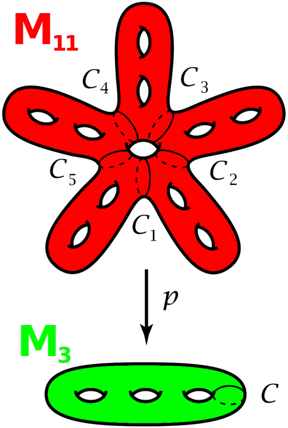
x is identified with all $g.x$

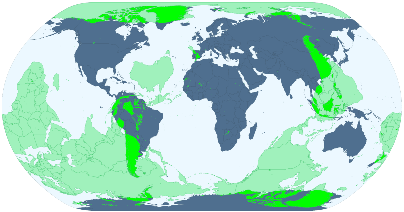
Such an action is a covering action (a good action) if

$$\forall x \in X \exists \text{ open neighborhood } U : g_1.U \cap g_2.U = \emptyset \text{ unless } g_1 = g_2$$

- The quotient of a covering action $p \colon X \to X/G$ is a covering
- If X is additionally path-connected and locally path-connected, then $G \cong \pi_1(X/G)/p_*(\pi_1(X))$
- Special cases of good actions are Deck transformations: $f \in \mathrm{Homeo}(\tilde{X})$ with $p \circ f = p$ for $p \colon \tilde{X} \to X$

Computing π_1





- Left case. $G = \mathbb{Z}$ acts by translation and $\pi_1(\mathbb{R}) \cong 1 \Rightarrow \pi_1(S^1) \cong \mathbb{Z}$
- Middle case. $\pi_1(M_{11})$ is an index 5 normal subgroup of $\pi_1(M_3)$
- Right case. $G = \mathbb{Z}/2\mathbb{Z}$ acts by antipodes and $\pi_1(S^2) \cong 1 \Rightarrow \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$

Thank you for your attention!

I hope that was of some help.