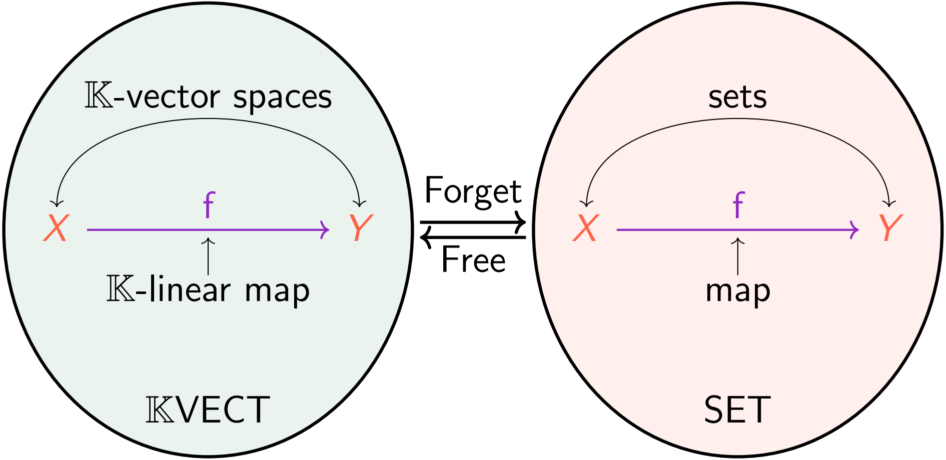
What is...the adjoint functor theorem?

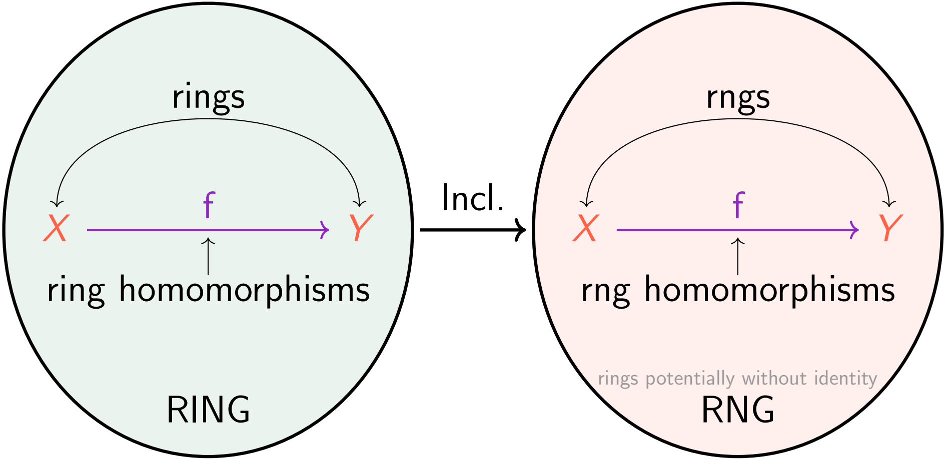
Or: Identities are free!

Adjoints preserve limits



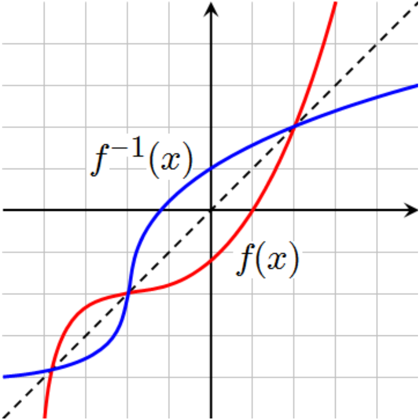
- ($Free$, $Forget$) adjunction preserves (colimit,limits)
- This is true in general, *i.e.* adjoint functors are (co)continuous
- Question What about the converse?

Testing (co)limits



- Does Inclusion:RING→RNG have adjoints? “Is adding units possible?”
- Inclusion($\mathbb{Z}$)$\cong \mathbb{Z}$ is not initial — No right adjoint exists
- Inclusion(0)$\cong$ 0 is terminal — Left adjoint? We can’t tell

Enough data?



- With enough data given one can construct f^{-1} from f
- RING is complete and Inclusion:RING→RNG preserves all limits
- We should be able to construct the left adjoint from the given data!?
- f^{-1}/adjoints do not always exist Expect extra condition!

For completeness: A formal statement

Given $G\colon C \to D$, assume that

- C is complete
- G is continuous
- The SSC holds, *i.e.* $\forall Y \in D\ \exists\ (f_i\colon Y \to G(X_i))_{i\in I}$ such that every $f\colon Y \to G(X)$ can be written as a composite (for some g_i):

$$\begin{array}{ccc} Y & \xrightarrow{f_i} & G(X_i) \\ & \searrow^{f} & \downarrow{\scriptstyle G(g_i)} \\ & & G(X) \end{array}$$

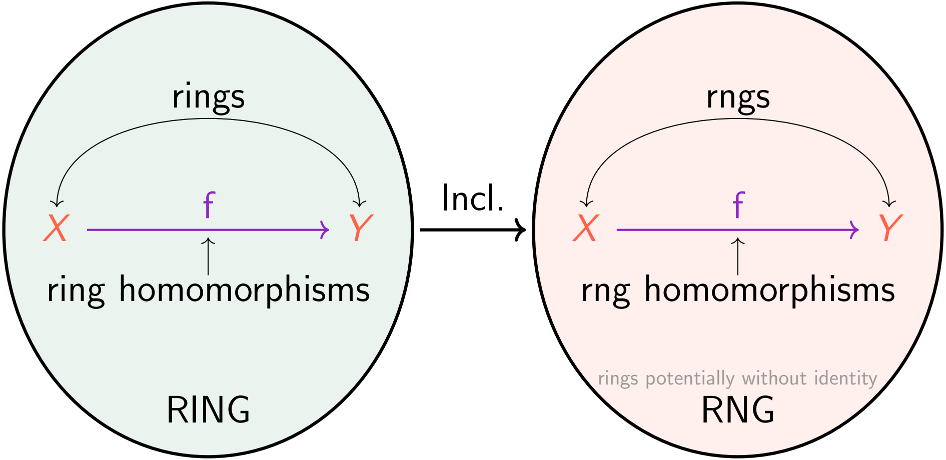
Then G has a left adjoint

- To avoid confusion, having a left adjoint means G is a right adjoint
- The above can be restated as: $G\colon X \to Y$ with complete domain, then

(G has a left adjoint) $\Leftrightarrow$ (G is continuous and satisfies SSC)

- There is a dual statement for the existence of right adjoints

Back to RGN



- Assume we would not know that adjoining identities works
- The adjoint functor theorem shows that Inclusion has a left adjoint F
- Define $F(X)$ as "universal" way to adjoin an identity

Thank you for your attention!

I hope that was of some help.