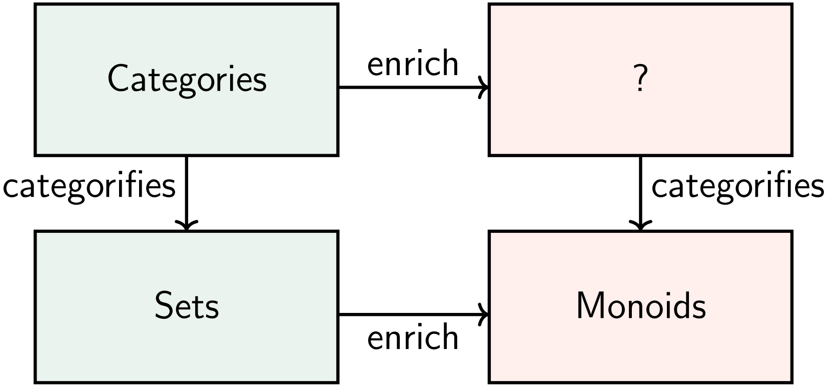
What are...monoidal categories?

Or: Monoids categorified

Filling in the questions mark



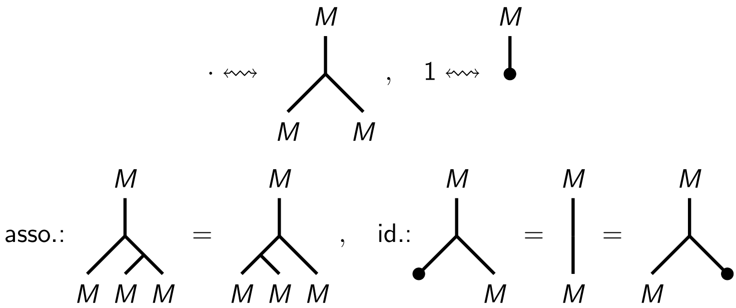
- Categories categorify sets
- Monoids are enriched sets
- What completes the square?

What we do not want!

Group-like structures					
	Totality$^{\alpha}$	Associativity	Identity	Invertibility	Commutativity
Semigroupoid	Unneeded	Required	Unneeded	Unneeded	Unneeded
Small category	Unneeded	Required	Required	Unneeded	Unneeded
Groupoid	Unneeded	Required	Required	Required	Unneeded
Magma	Required	Unneeded	Unneeded	Unneeded	Unneeded
Quasigroup	Required	Unneeded	Unneeded	Required	Unneeded
Unital magma	Required	Unneeded	Required	Unneeded	Unneeded
Semigroup	Required	Required	Unneeded	Unneeded	Unneeded
Loop	Required	Unneeded	Required	Required	Unneeded
Inverse semigroup	Required	Required	Unneeded	Required	Unneeded
Monoid	Required	Required	Required	Unneeded	Unneeded
Commutative monoid	Required	Required	Required	Unneeded	Required
Group	Required	Required	Required	Required	Unneeded
Abelian group	Required	Required	Required	Required	Required

- Monoids also appear *e.g.* via monads in categories
- Monads are not categorifications of monoids; just “similar in nature”
- A categorification should have two operations

Monoids



A monoid $(M, \cdot)$ consists of

- A set M
- A multiplication $\cdot\colon M \times M \to M$ (write $ab = a \cdot b$)
- A unit $1 \in M$

such that

- $\cdot$ is associative $a(bc) = (ab)c$
- $\cdot$ is unital $1a = a = a1$

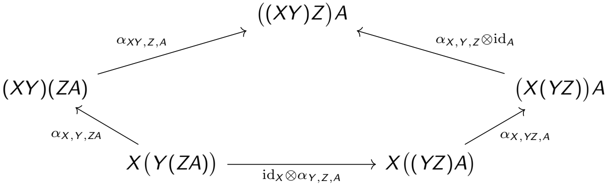
For completeness: A formal definition

A **monoidal category** $(C, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ consists of

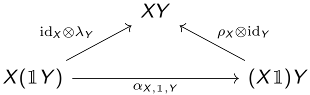
- A category C
- A bifunctor $\otimes\colon C \times C \to C$ (write $XY = X \otimes Y$)
- A unit object $\mathbb{1} \in C$
- Natural isomorphisms $\alpha_{X,Y,Z}\colon X(YZ) \to (XY)Z$, $\lambda_X\colon \mathbb{1}X \to X$, $\rho_X\colon X\mathbb{1} \to X$

such that

(a) the ⬠ *equality* holds, *i.e.* we have commuting diagrams



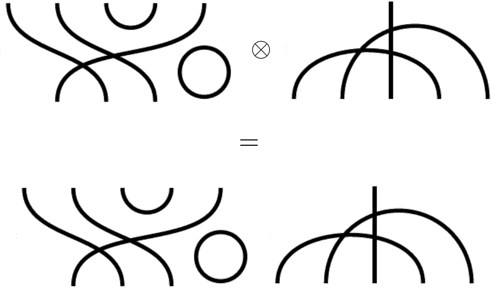
(b) the △ *equality* holds, *i.e.* we have commuting diagrams



Some examples

Name	Objects	Arrows	$\otimes$
SET	Sets	Maps	$\times$
CAT	Categories	Functors	$\times$
1COB	0-manifolds	1-manifolds	See below
nCOB	(n-1)-manifolds	n-manifolds	Similarly as below
$\mathbb{K}$VECT	$\mathbb{K}$-vector spaces	$\mathbb{K}$-linear map	$\otimes$
$\mathbb{K}$VECT	$\mathbb{K}$-vector spaces	$\mathbb{K}$-linear map	$\oplus$

Most diagrammatic categories are monoidal via juxtaposition:



Thank you for your attention!

I hope that was of some help.