What are...2-categories?

Or: Categories in categories

2-categories were around from the very beginning

GENERAL THEORY OF NATURAL EQUIVALENCES

BY

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CONTENTS

- Eilenberg–Mac Lane introduced categories in $\sim$1945
- The main focus were already functors and nat trafos
- What is a good home for nat trafos?

Higher dimensions

The ladder of categories

$\vdots$

forget 3-arrows ↓ / ↑ add 3-arrows

$2-\text{categories} = $ "arrows between arrows"

forget 2-arrows ↓ / ↑ add 2-arrows

$1-\text{categories} = $ usual categories

forget arrows ↓ / ↑ add arrows

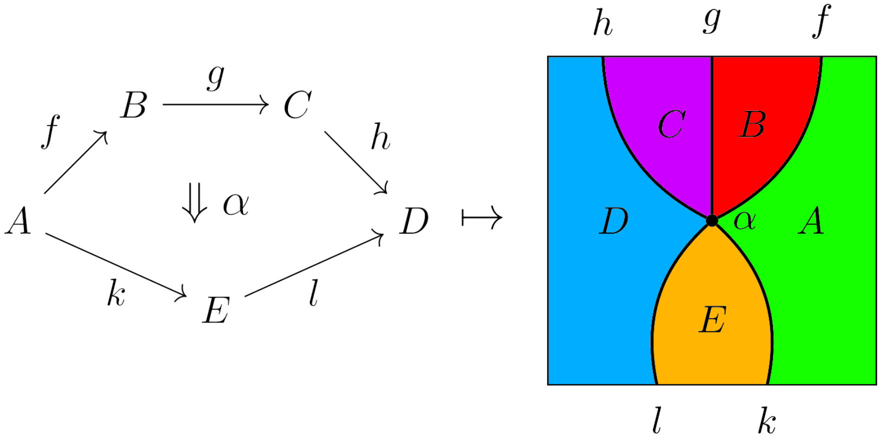
$0-\text{categories} = $ sets, vector spaces

size ↓ / ↑ internal structure

$-1-\text{categories} = $ cardinals, truth values

- A prototypical example of a set is the set of numbers
- A prototypical example of a category is the category of sets
- A prototypical example of a 2-category is the category of categories

String diagram



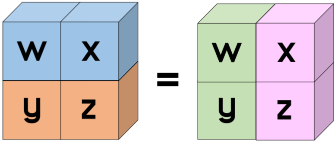
- **CAT**: objects are categories, arrows are functors, 2-arrows are nat trafos
- Slogan A 2-category should be defined to admit string diagrams
- Warning, this is Poincaré dual to the more standard illustration

For completeness: A formal definition

A 2-category C is a category C enriched in categories, vaguely meaning:

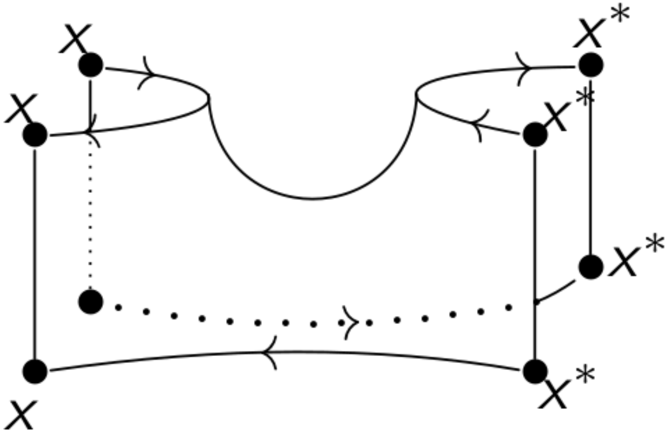
- It has objects and arrows
- Each $\hom_C(X, Y)$ is a category rather than a set
- Thus, there are also 2-arrows being "arrows between arrows"
- Arrows can be composed in one direction $\circ$, 2-arrows in two directions $\circ_v, \circ_h$
- Everything is up to natural sets of axioms, most notably:

$$(w \circ_h x) \circ_v (y \circ_h z) = (w \circ_v y) \circ_h (x \circ_v z) \quad \leftrightsquigarrow$$



- Example The 2-category of categories **CAT**
- Example Every monoidal category is a 2-category with one object
- There is also a weaker notion, but then again a strictification result

Cobordisms again



- **2COB**: objects are points, arrows are lines, 2-arrows are surfaces
- All axioms in this 2-category are visually clear ;-)

Thank you for your attention!

I hope that was of some help.