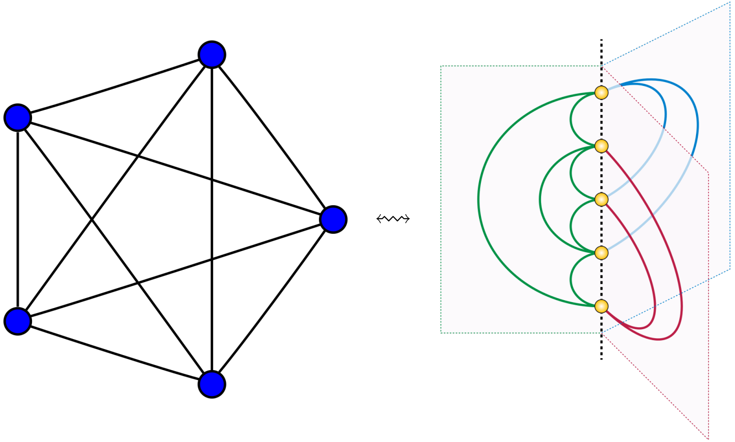
What is...the chromatic number of a surface?

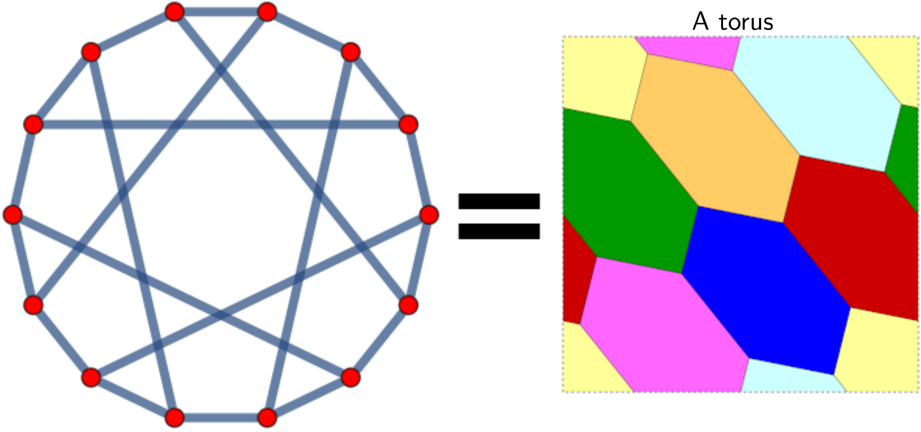
Or: Graphs on surfaces

Graph embeddings



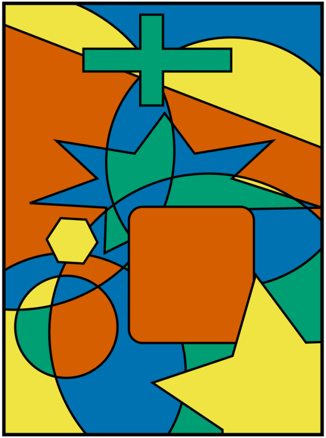
- The book embedding shows that any graph embeds into $\mathbb{R}^3$
- For a fixed surface S this is however not true
- Question What can we say about graph embeddings in S?

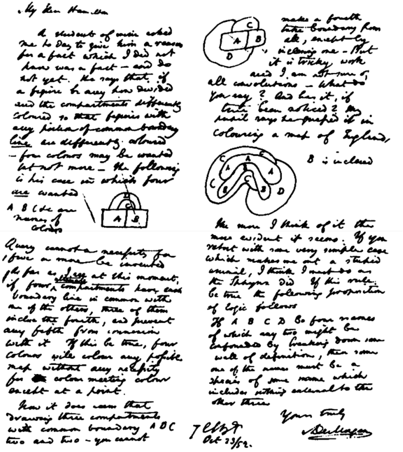
Graphs on surfaces



- Study embeddings of graphs on surfaces
- To find the minimal surface a given graph embeds into is very hard
- Let us do the opposite: fix a surface and consider graph embedded into it

Coloring embedded graphs





My dear Hamilton

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted

A B C &c are names of colours

Query cannot a necessity for five or more be invented. As far as I see at this moment, if four ultimate compartments have each boundary line in common with one of the others, three of them inclose the fourth, and prevent any fifth from connexion with it. If this be true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.

Now it does seem that drawing three compartments with common boundary ABC two and two — you cannot make a fourth take boundary from all, except by inclosing one — But it is tricky work and I am not sure of all convolutions — what do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England.

B is inclosed

The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did. If this rule be true the following proposition of logic follows

If A B C D be four names of which any two might be confounded by breaking down some wall of definition, then some one of the names must be a species of some name which includes nothing external to the other three.

Yours truly
A De Morgan

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Oct 23/52.

- Every graph embedded in a surface S admits the notion of faces
- Question What is the minimal number $C(S)$ of colors needed to color any graph in S?
- The four color theorem is the most famous instance of this

For completeness: A formal statement

For a closed connected surface $S \neq$ (S^2 or Klein bottle) we have

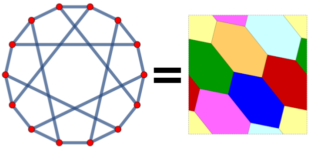
$$C(S) = \left\lfloor \frac{1}{2}\left(7 + \sqrt{49 - 24\chi(S)}\right) \right\rfloor$$

where $\chi(S)$ is the Euler characteristic of S

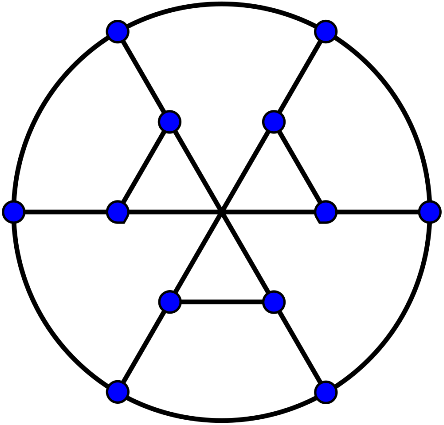
- Heawood's number is almost always perfect:

Surface	Heawood's bound	real $C(S)$
S^2	6	4
$\mathbb{K}$	7	6
$S \neq S^2, \mathbb{K}$	$c = \left\lfloor \frac{7+\sqrt{49-24\chi(S)}}{2} \right\rfloor$	c

- For a torus $\chi(S) = 0$ so we get $C(S) = 7$:



The two missing cases



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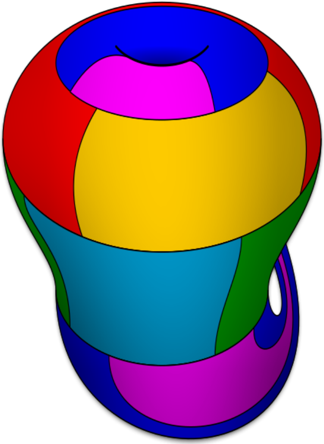

- For S^2 "=" plane the correct number is $C(S) = 4$ and this is really difficult to prove – four color theorem
- For the Klein bottle the correct number is $C(S) = 6$ and this due to Franklin $\sim$1930

Thank you for your attention!

I hope that was of some help.