What are...homology spheres?

Or: Spheres, but not really...

Homology counts holes

The torus T and the solid torus T^s

$$: \begin{cases} \dim H_0 = 1 \\ \dim H_1 = 2 \\ \dim H_2 = 1 \end{cases}$$

$$: \begin{cases} \dim H_0 = 1 \\ \dim H_1 = 1 \\ \dim H_2 = 0 \end{cases}$$

- A zero dimensional hole $\dim H_0$ is a connected component
- A one dimensional hole $\dim H_1$ is the number of necklaces you can put it on
- A two dimensional hole $\dim H_2$ is the number of plugs needed to inflate it

Eric Weisstein "A hole in a mathematical object is a topological structure which prevents the object from being continuously shrunk to a point."

I use homology is a black box – it is some algebraic datum associated to a space

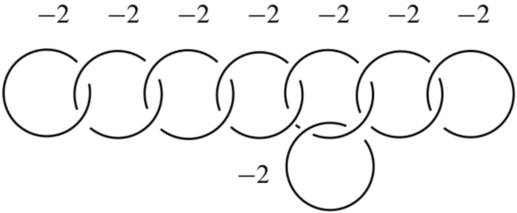
Spheres algebraically

$S^2 =$  $, H^*(S^2) \leftrightsquigarrow 1 + t$

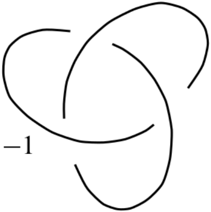
- S^n is among the easiest nontrivial spaces; its homology is essentially trivial
- For dim 2: "homology of manifold X = homology of S^2" $\Leftrightarrow$ "$X = S^2$"
- Question What about 3d?

Poincaré's example

$P =$



$P =$



- Surgery around the above knots gives Poincaré's homology sphere P
- P is very far from a sphere, *e.g.* $|\pi_1(S^3)| = 1$ but $|\pi_1(P)| = 120$

For completeness: A formal statement

An oriented 3mfd is...

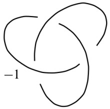
(i) ...a *R* homology sphere if it has the same R homology as S^2

(ii) ...an homology sphere if it has the same $\mathbb{Z}$ homology as S^2

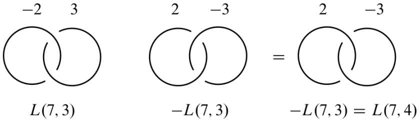
(iii) ...a rational homology sphere if it has the same $\mathbb{Q}$ homology as S^2

Theorem These beasts exist!

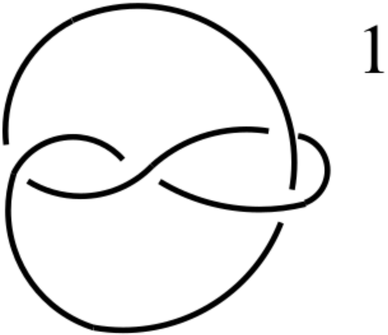
- Surgery on a knot with framing ± 1 gives a homology sphere



- Lens spaces $L(p, q)$ are rational homology spheres



A family of these beasts

$\Sigma(2,3,7) =$ 

- Take $p, q, r \in \mathbb{N}$ pairwise relatively prime
- The intersection $\Sigma(p,q,r)$ of a small 5 sphere around 0 with $x^p + y^q + z^r = 0$ is a homology sphere Seifert homology spheres / Brieskorn manifolds
- $\Sigma(2,3,5)$ is the Poincaré sphere
- There is a very explicit way of defining these – will be discussed next time

Thank you for your attention!

I hope that was of some help.