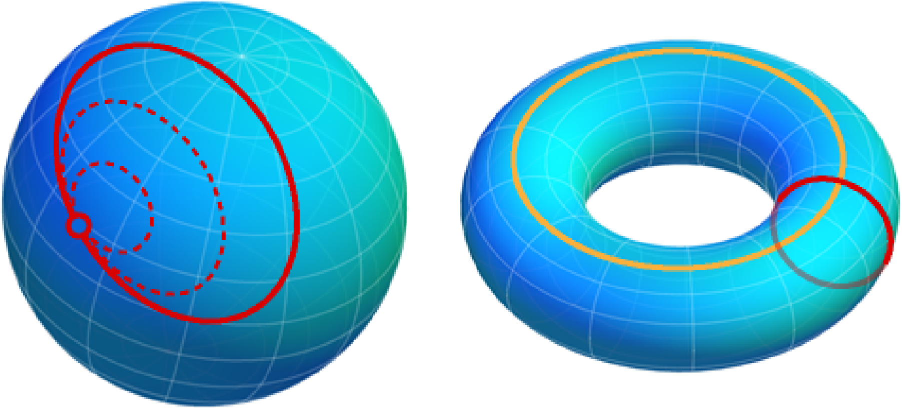
What is...the h-cobordism theorem in action?

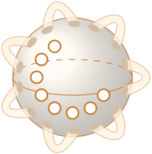
Or: Towards the Poincaré conjecture

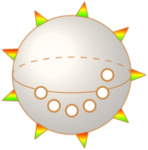
Reminder: The Poincaré conjecture (PC)

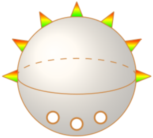


- PC Every homotopy n sphere is homeomorphic to the n sphere
- For $n = 3$ this is "Every closed 3mfd with trivial fundamental group is homeomorphic to the 3 sphere
- We will sketch a proof for $n \neq 3, 4$ using e.g. the h-cobordism theorem

Let us prove the PC!

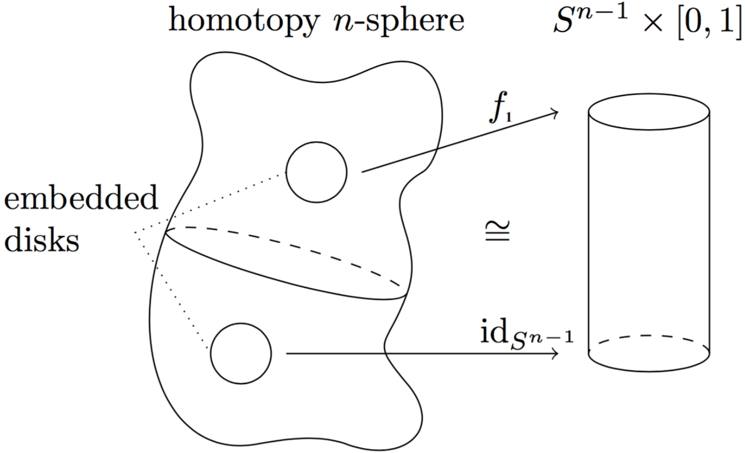
$\#^{8}\mathbb{D}^{2} \# \#^{7}\mathbb{T} \cong$ 

$\#^{6}\mathbb{D}^{2} \# \#^{9}\mathbb{P}^{2} \cong$ 

$\#^{3}\mathbb{D}^{2} \# \#^{2}\mathbb{T} \# \#^{3}\mathbb{P}^{2} \cong$  $\cong$ 

- **Dims 1 and 2** In these cases we can use classification results
- **Dims 3 and 4** Too hard for me
- **Dims ≥ 5** The h-cobordism theorem applies

Poking twice

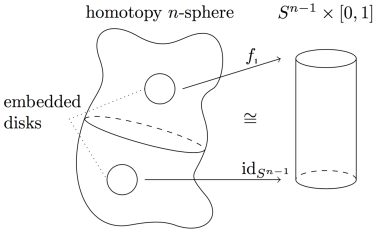


- Take two embedded disjoint discs $D_1, D_2 \hookrightarrow M$ in a homotopy n sphere M
- Poke and get a cobordism $W = M \setminus int(D_1) \coprod int(D_2)$
- "Classic algebraic topology methods" show that W is an h-cobordism

For completeness: A formal statement

Since W is an h-cobordism we get

$$(W; \partial D_1, \partial D_2) \cong (\partial D_1 \times [0,1], \partial D_1 \times \{0\}, \partial D_1 \times \{1\})$$



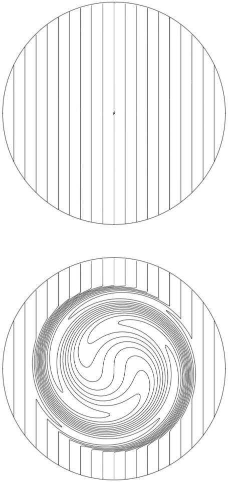
Note here that $\partial D_1 \times [0,1] \cong S^{n-1} \times [0,1]$, so we get $f\colon S^{n-1} \xrightarrow{\cong} S^{n-1}$

$f\colon S^{n-1} \xrightarrow{\cong} S^{n-1}$ extends to a homeomorphism $F\colon D^n \xrightarrow{\cong} D^n$ (next slide)

Fill in the discs and use F to get that M is homeomorphic to S^n

- Note that this only works for $n \geq 5$ as the h-cobordism theorem requires that
- For $n = 3, 4$ different techniques are needed

Alexander's trick



- Two homeomorphism $f, g\colon D^n \to D^n$ that agree on $\partial D^n = S^{n-1}$ are isotopic
- Or backwards Knowing $f\colon S^{n-1} \xrightarrow{\cong} S^{n-1}$ gives a unique $F\colon D^n \xrightarrow{\cong} D^n$

Thank you for your attention!

I hope that was of some help.