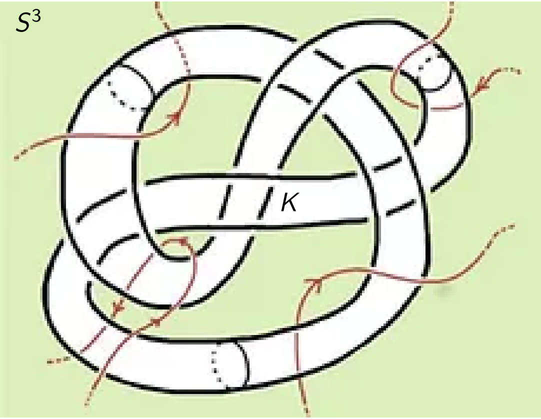
What are...knot groups?

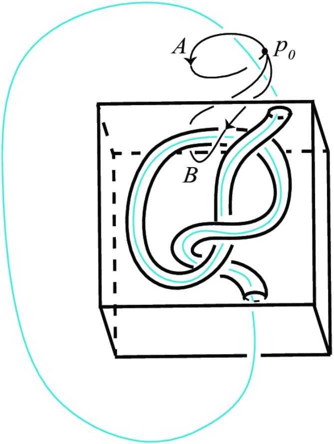
Or: Never forget the complement

Everything except the knot



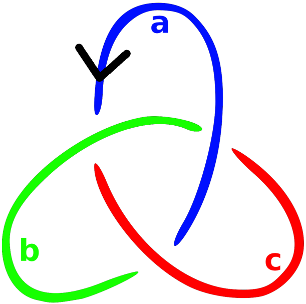
- Think of a knot $K \subset S^3$ as an embedding of a solid torus Thickening
- The knot complement is the 3d space $X_K = S^3 \setminus \text{int}(K)$
- Example X_{unknot} is a solid torus (think of $\mathbb{R}^3$ without the z axis)

The knot group topologically



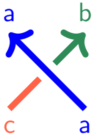
- The knot group $\pi_1(K)$ is $\pi_1(X_K)$
- $\pi_1(K)$ is thus generated by loops in the knot complement
- $\pi_1(K)$ is a strong knot invariant

The knot group algebraically

$K =$ 

$$\pi_1(K) = \langle a, b, c | ab = ca, bc = ab, ca = bc \rangle \cong \langle a, b | aba = bab \rangle$$

- Define $\pi_1(K)$ (of a projection) as generated by arcs in the projection modulo

$\rightsquigarrow ab = ca,$


$\rightsquigarrow ac = ba$

- Wirtinger The algebraic $\pi_1(K)$ is the topological $\pi_1(K)$ up to isomorphism

For completeness: A formal statement

Define the knot group topologically or algebraically

The knot group is a knot invariant

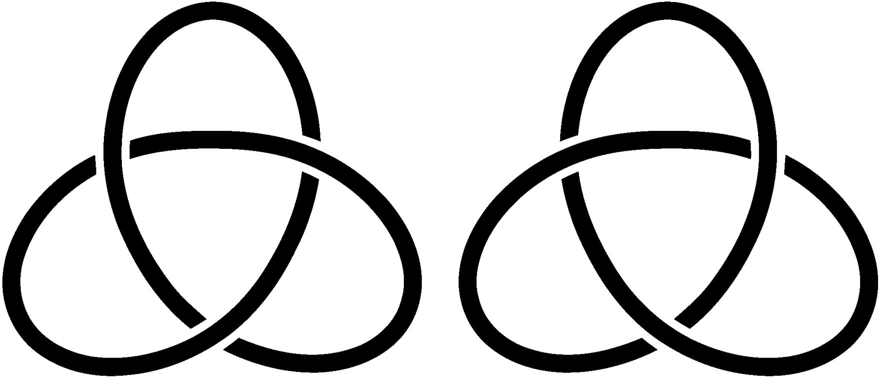
Both definitions agree up to isomorphism

- The topological incarnation is clearly a knot invariant but it is unclear how to compute it
- The algebraic incarnation is clearly computable but it is unclear why it is a knot invariant
- Hence, identifying them is key
- The knot complement X_K itself is even better:

"Two knots/mirrors are the same $\Leftrightarrow$ their knot complements are the same"

Warning: this is not true for links

Left = right-handed trefoil? No idea...



- The left-handed trefoil has knot group $\langle a, b | bab = aba \rangle$ (the braid group B_3)
- The right-handed trefoil has knot group $\langle a, b | bab = aba \rangle$ (the braid group B_3)
- Thus, we still can't tell them apart

Thank you for your attention!

I hope that was of some help.