**What is...a knot homology?**

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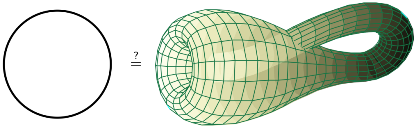
Or: Vector spaces, not numbers

## Homology, Hilbert–Poincaré, Euler

| | | |
|---|---|---|
| An honest module | (co)homology $H^*$ | Khovanov homology $H^*$ |
| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| Graded dimensions | Hilbert–Poincaré polynomial $P$ | Khovanov polynomial $P$ |
| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $t = -1$ | Euler $\chi$ | Jones polynomial $V$ |




- $H_*$ distinguishes $S^1$ from $K$

$$H_*(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad H_*(K) \cong \mathbb{Z} \oplus (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$$

- $P$ does not distinguish $S^1$ from $K$

$$P(S^1) = P(K) = 1 + t$$

- $\chi$ does not distinguish $S^1$ from $K$

$$\chi(S^1) = \chi(K) = 0$$

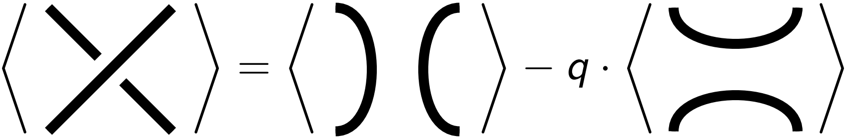
Knot homology (Khovanov, knot Floer, more...) to link polynomial (Jones, Alexander, more...) = homology to Euler characteristic

## Kauffman skein calculus rescaled

(i) $\langle \emptyset \rangle = 1$ Normalization

(ii) $\langle \bigcirc \cup L \rangle = (q + q^{-1}) \cdot \langle L \rangle$ Pulling out circles

(iii) Kauffman Skein



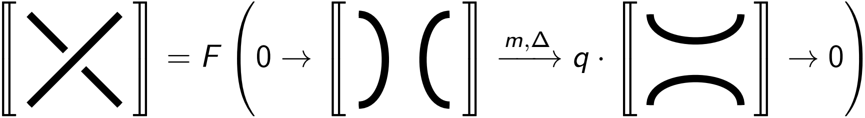
- Empty knot is normalized to 1
- Circle = “number”
- Kauffman skein relation = linear relation among “numbers”

# Khovanov–Bar Natan skein calculus

(i) $[\![\emptyset]\!] = \mathbb{Q}$ Normalization

(ii) $[\![\bigcirc \cup L]\!] = V \otimes [\![L]\!]$ with $V$ of grdim $q + q^{-1}$ Pulling out circles

(iii) Khovanov–Bar Natan Skein




$F$ = certain operation on chain complexes

- Empty knot is normalized to $\mathbb{Q}$
- Circle = vector space of grdim $q + q^{-1}$
- Khovanov–Bar Natan skein relation = relation in chain complexes
- The crucial $m, \Delta$ will reappear later - for now: they exist

# For completeness: A formal statement

**Up to normalization ⟦_⟧ is a knot invariant**

taking values in chain complexes

- Taking homology of gives a link invariant in gr VS called **Khovanov homology**
- We have the categorification picture

| | | |
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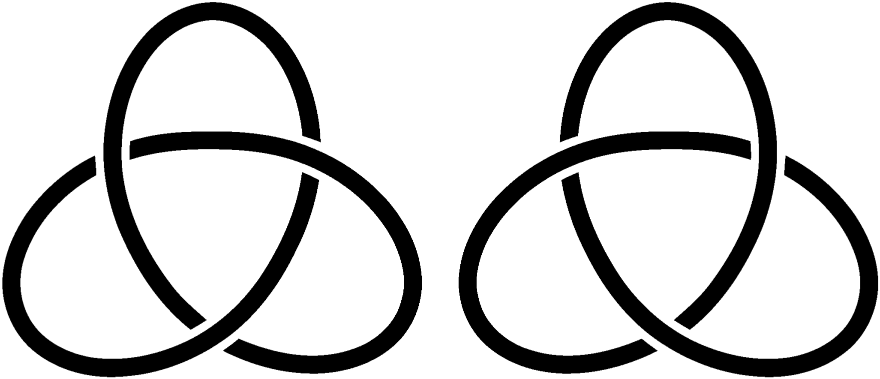
| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $t = -1$ | Euler $\chi$ | Jones polynomial $V$ |

| $j$ \ $r$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| 5 | | | | | 1,1,1 |
| 3 | | | | 0,1,0 | 0,1,1 |
| 1 | | | 1,1,1 | 1,1,0 | |
| $-1$ | | 1,1,1 | 1,1,2 |  | |
| $-3$ | 0,1,0 | 0,1,1 | 0,0,2 | | |
| $-5$ | 1,1,0 | | 0,0,2 | | |
| $< -5$ | | | 0,0,2 | | |

## Left = right-handed trefoil? Strongly no!



- The left-handed trefoil has Khovanov polynomial $q^3 + q + q^9 t^3 + q^5 t^2$
- The right-handed trefoil has Khovanov polynomial $\frac{1}{q^3} + \frac{1}{q} + \frac{1}{q^9 t^3} + \frac{1}{q^5 t^2}$
- Thus, they are different – and Khovanov homology detects them as a pair

**Thank you for your attention!**

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I hope that was of some help.