

What is...a knot homology?

Or: Vector spaces, not numbers

Homology, Hilbert–Poincaré, Euler

An honest module



Graded dimensions



$$t = -1$$

(co)homology H^*



Hilbert–Poincaré polynomial P



Euler χ

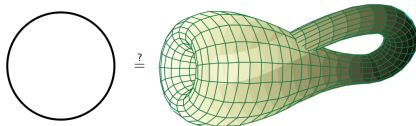
Khovanov homology H^*



Khovanov polynomial P



Jones polynomial V



- ▶ H_* distinguishes S^1 from K

$$H_*(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad H_*(K) \cong \mathbb{Z} \oplus (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$$

- ▶ P does not distinguish S^1 from K

$$P(S^1) = P(K) = 1 + t$$

- ▶ χ does not distinguish S^1 from K

$$\chi(S^1) = \chi(K) = 0$$

Knot homology (Khovanov, knot Floer, more...) to link polynomial (Jones, Alexander, more...) = homology to Euler characteristic

Kauffman skein calculus rescaled

(i) $\langle \emptyset \rangle = 1$ Normalization

(ii) $\langle \bigcirc \cup L \rangle = (q + q^{-1}) \cdot \langle L \rangle$ Pulling out circles

(iii) Kauffman Skein



The diagram illustrates the Kauffman Skein relation. On the left, a crossing of two strands is enclosed in large angle brackets. This is set equal to the sum of two terms. The first term is a pair of strands that are parallel on the left and then curve to meet at a point on the right, forming a shape like a closing parenthesis ')'. The second term is a pair of strands that are parallel on the left and then curve to meet at a point on the right, forming a shape like an opening parenthesis '('. This second term is multiplied by a minus sign and the variable 'q'. The final term is a pair of strands that are parallel on the left and then curve away from each other on the right, forming two separate arcs, one above and one below the space between the strands.

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- ▶ Empty knot is normalized to 1
 - ▶ Circle = “number”
 - ▶ Kauffman skein relation = linear relation among “numbers”

Khovanov–Bar Natan skein calculus

(i) $[\emptyset] = \mathbb{Q}$ Normalization

(ii) $[\bigcirc \cup L] = V \otimes [L]$ with V of $\text{grdim } q + q^{-1}$ Pulling out circles

(iii) Khovanov–Bar Natan Skein

$$\left[\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right] = F \left(0 \rightarrow \left[\begin{array}{c} \text{) } \\ \text{(} \end{array} \right] \xrightarrow{m, \Delta} q \cdot \left[\begin{array}{c} \text{) } \\ \text{(} \end{array} \right] \rightarrow 0 \right)$$

F = certain operation on chain complexes

- ▶ Empty knot is normalized to \mathbb{Q}
- ▶ Circle = vector space of $\text{grdim } q + q^{-1}$
- ▶ Khovanov–Bar Natan skein relation = relation in chain complexes
- ▶ The crucial m, Δ will reappear later - for now: they exist

For completeness: A formal statement

Up to normalization $[\![\]\!]$ is a knot invariant

taking values in chain complexes

- ▶ Taking homology of gives a link invariant in gr VS called **Khovanov homology**
- ▶ We have the categorification picture

An honest module



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(co)homology H^*



Hilbert–Poincaré polynomial P



Euler χ


Khovanov homology H^*



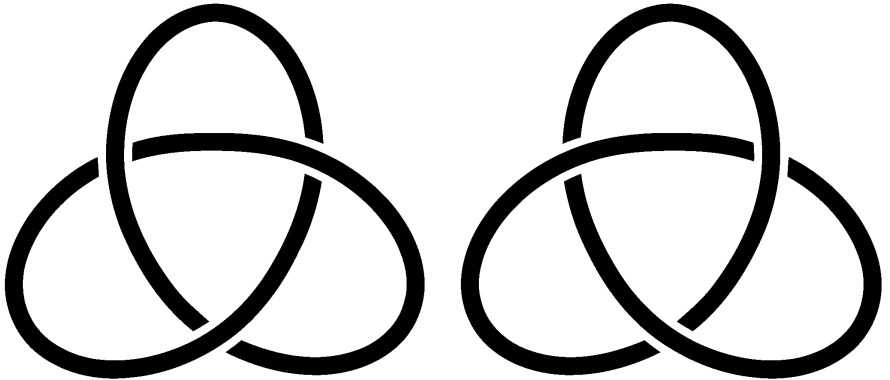
Khovanov polynomial P



Jones polynomial V

$j \backslash r$	-2	-1	0	1	2
5					1,1,1
3				0,1,0	0,1,1
1			1,1,1	1,1,0	
-1		1,1,1	1,1,2		
-3	0,1,0	0,1,1	0,0,2		
-5	1,1,0		0,0,2		
< -5			0,0,2		

Left = right-handed trefoil? Strongly no!



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- ▶ The left-handed trefoil has Khovanov polynomial $q^3 + q + q^9 t^3 + q^5 t^2$
 - ▶ The right-handed trefoil has Khovanov polynomial $\frac{1}{q^3} + \frac{1}{q} + \frac{1}{q^9 t^3} + \frac{1}{q^5 t^2}$
 - ▶ Thus, they are different – and Khovanov homology detects them as a pair

Thank you for your attention!

I hope that was of some help.