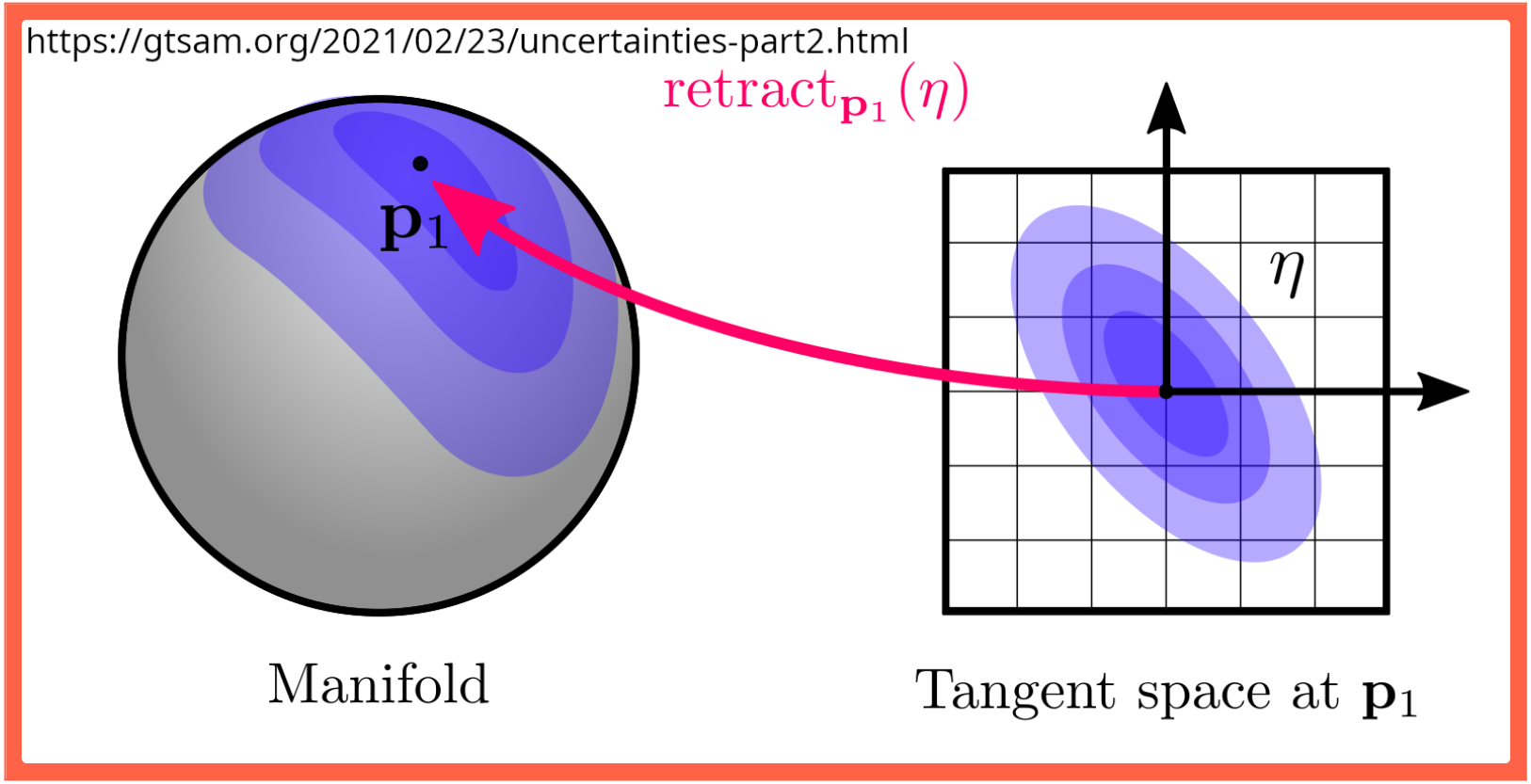
Lie theory - part 10

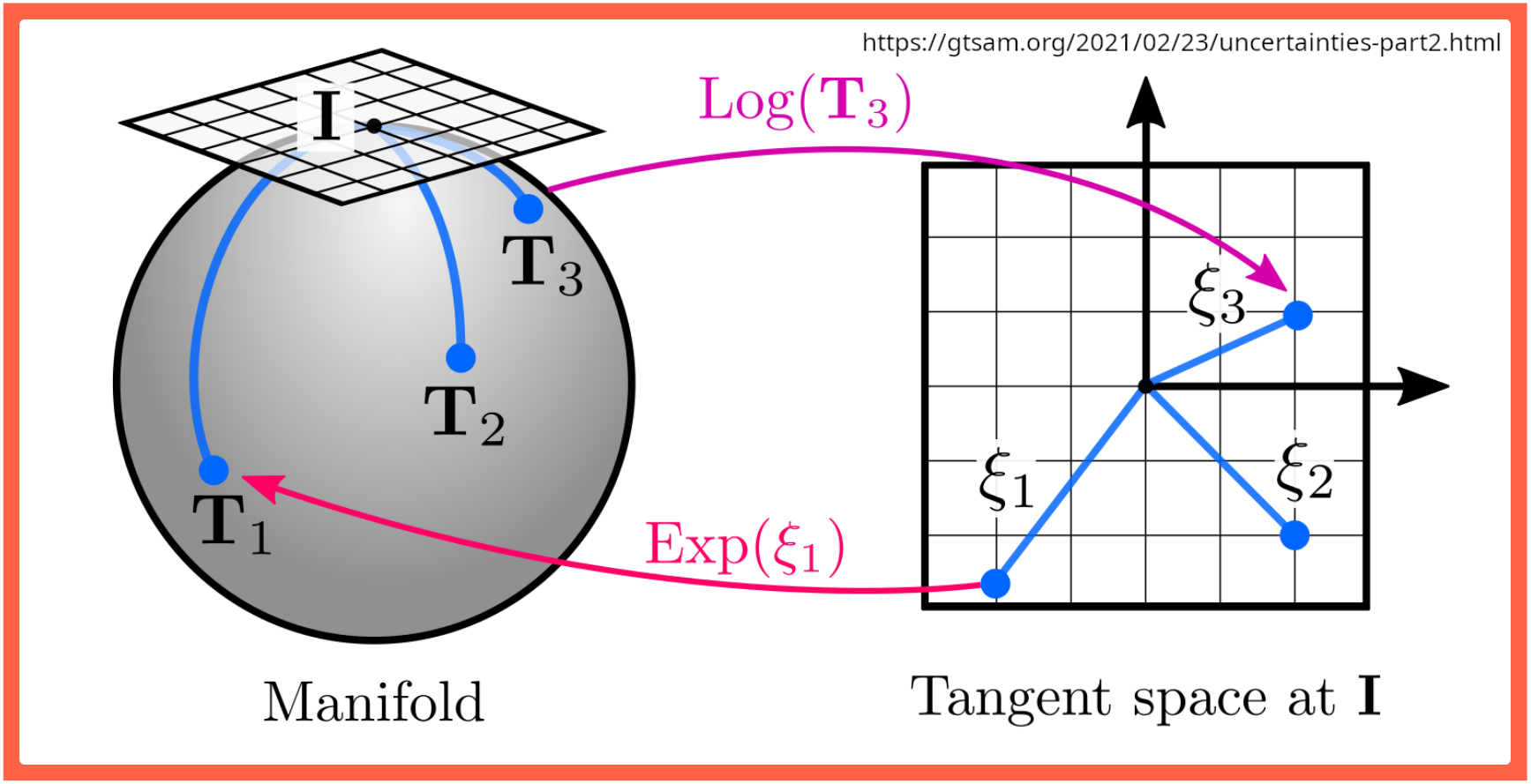
Or: The exponential map in more details

From infinitesimal data to actual motion



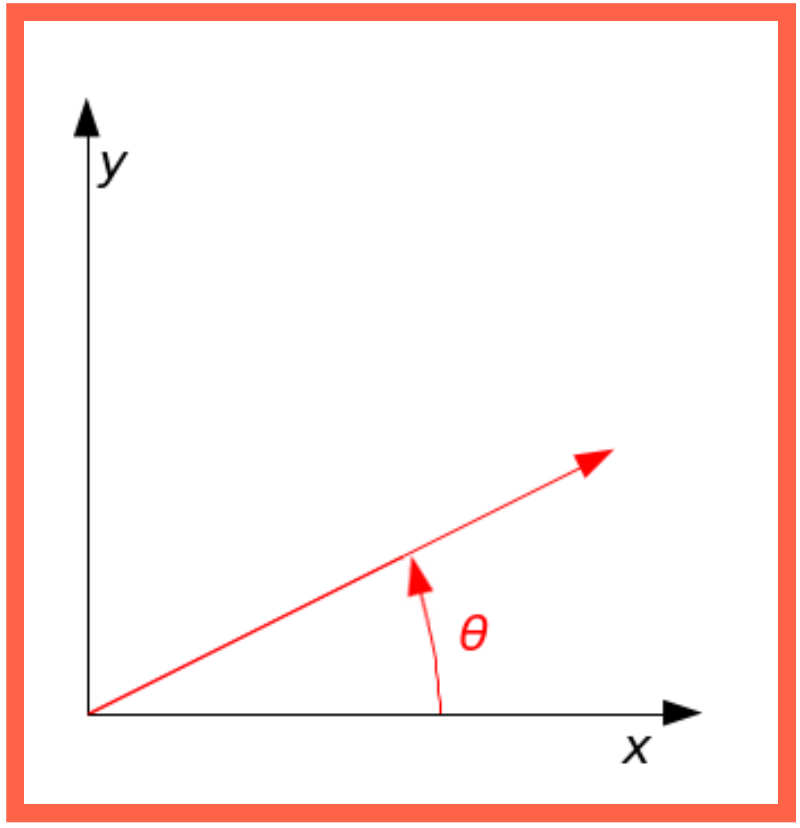
- **Definition** For matrices, $\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$
- **Dynamic meaning** The curve $t \mapsto e^{tA}$ is the flow generated by the infinitesimal direction A
- **Big picture** Lie algebras describe tiny motions; the exponential map turns them into nearby group elements

What the exponential map is (and is not)



- Local story Near $0 \in \mathfrak{g}$ and $e \in G$, the exponential map behaves like a smooth change of coordinates
- Not global It need not be injective and need not hit every element of G
- Moral The Lie algebra captures the geometry of the group close to the id

Examples: rotations

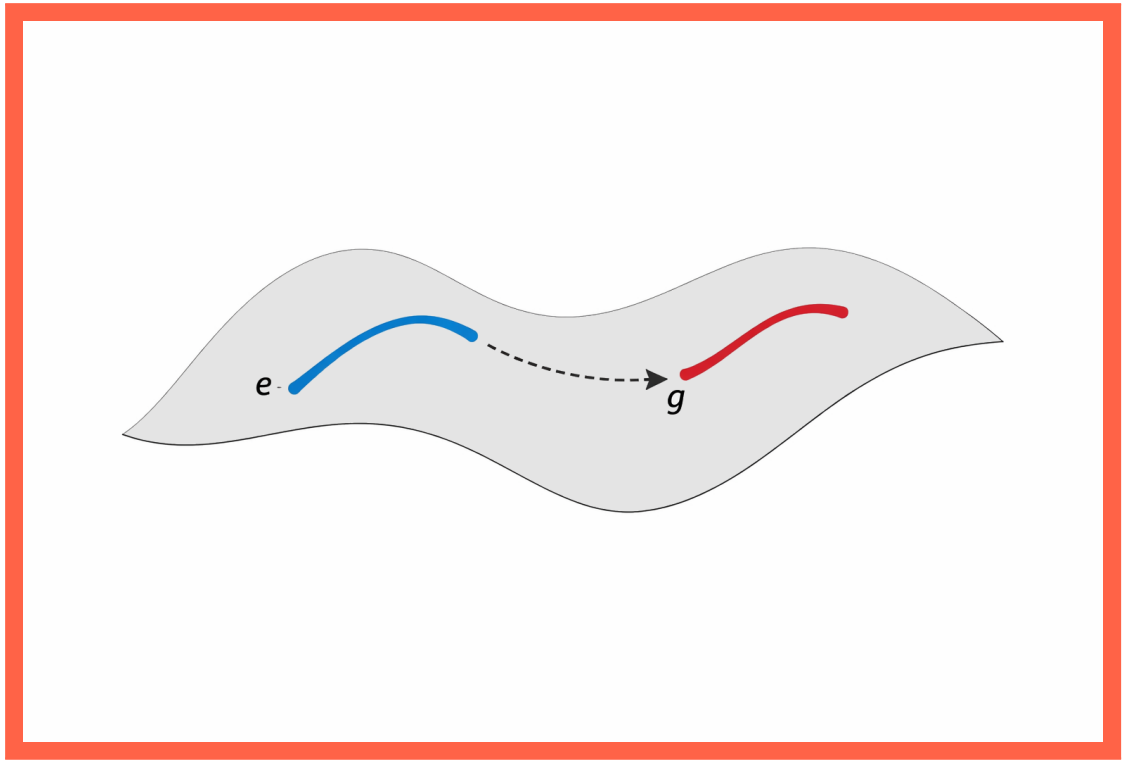


- Rotation example In $\mathfrak{so}(2)$, a skew-symmetric matrix exponentiates to an honest rotation matrix
- Rotation example For

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{so}(2), \qquad \exp(tA) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

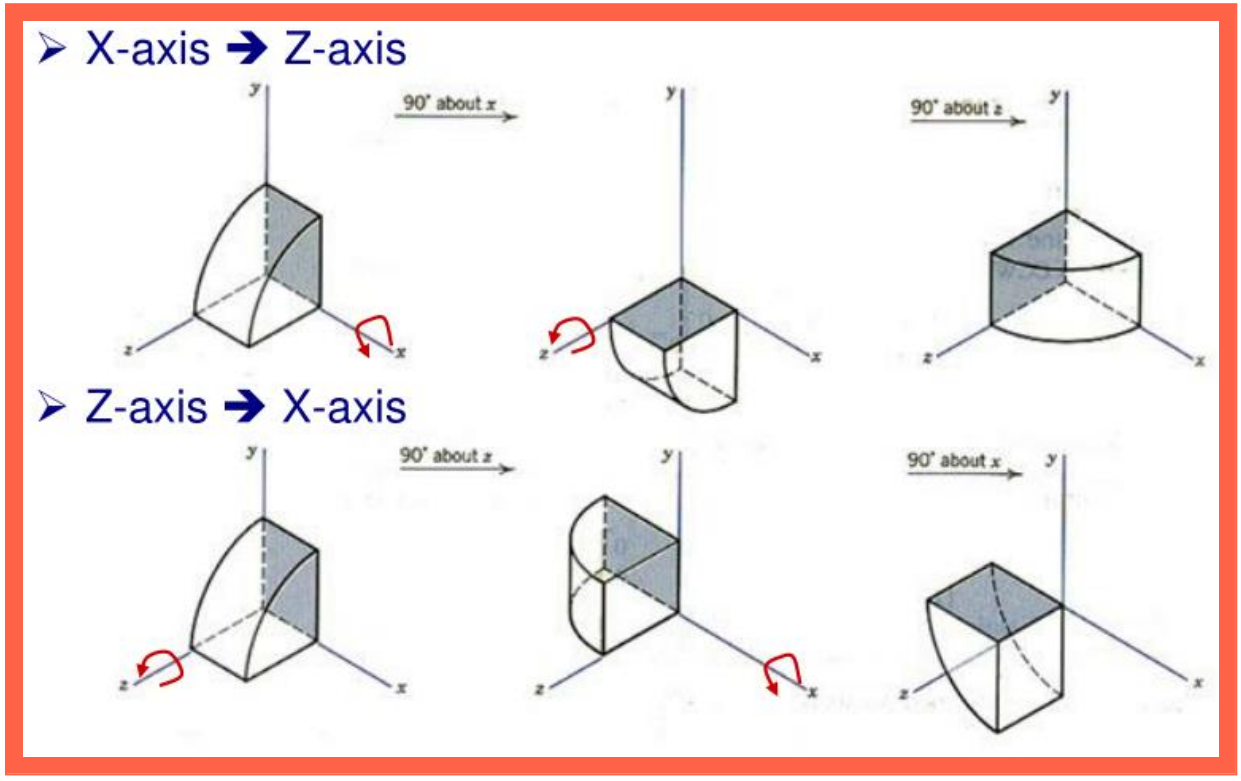
which is exactly rotation by angle t

Why the derivative at zero matters so much



- First-order term Since $\exp(tA) = I + tA + O(t^2)$, the tangent vector at $t = 0$ is exactly A
- Consequence The differential of exp at 0 is the identity map on $\mathfrak{g}$
- Translation This is why Lie algebra data really does encode the local geometry of the group

Why we care: products are harder than sums



- Easy case If A and B commute, then $e^{A+B} = e^A e^B$
- Hard case In general, multiplication in the group is more complicated than addition in the Lie algebra
- Preview Next: Baker–Campbell–Hausdorff explains the logarithm of a product near the identity

Thank you for your attention!

Next time: Baker–Campbell–Hausdorff