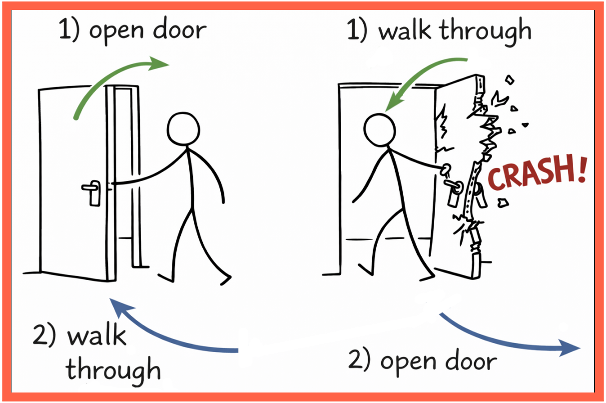
Lie theory - part 11

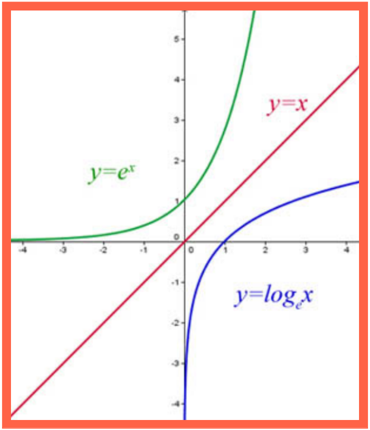
Or: Baker–Campbell–Hausdorff

Why Baker–Campbell–Hausdorff (BCH) exists at all



- **Big picture** In the Lie algebra we add, but in the Lie group we multiply: BCH compares these two worlds
- **Local move** Near the identity, we can take $e^X e^Y$ in the group and ask for its logarithm back in $\mathfrak{g}$
- **Output** BCH says that $\log(e^X e^Y)$ is built from X, Y, and their brackets

The easy case: when everything commutes



- Commuting case If $[X, Y] = 0$, then nothing exotic happens: $e^{X+Y} = e^X e^Y$
- Logarithm So in this friendly case, $\log(e^X e^Y) = X + Y$
- Moral BCH only becomes interesting once the infinitesimal directions stop commuting

The first correction terms already tell the story

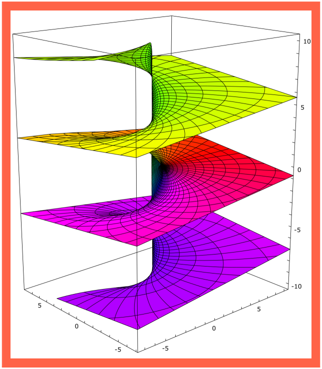
$$
\begin{aligned}
Z(X,Y) = \log(\exp X \exp Y) \\
&= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) \\
&\quad - \frac{1}{24}[Y,[X,[X,Y]]] \\
&\quad - \frac{1}{720}\left([Y,[Y,[Y,[Y,X]]]] + [X,[X,[X,[X,Y]]]]\right) \\
&\quad + \frac{1}{360}\left([X,[Y,[Y,[Y,X]]]] + [Y,[X,[X,[X,Y]]]]\right) \\
&\quad + \frac{1}{120}\left([Y,[X,[Y,[X,Y]]]] + [X,[Y,[X,[Y,X]]]]\right) \\
&\quad + \frac{1}{240}\left([X,[Y,[X,[Y,[X,Y]]]]]\right) \\
&\quad + \frac{1}{720}\left([X,[Y,[X,[X,[X,Y]]]]] - [X,[X,[Y,[Y,[X,Y]]]]]\right) \\
&\quad + \frac{1}{1440}\left([X,[Y,[Y,[Y,[X,Y]]]]] - [X,[X,[Y,[X,[X,Y]]]]]\right) + \cdots
\end{aligned}
$$

- First terms

$$\log(e^X e^Y) = X + Y + \tfrac{1}{2}[X,Y] + \ldots$$

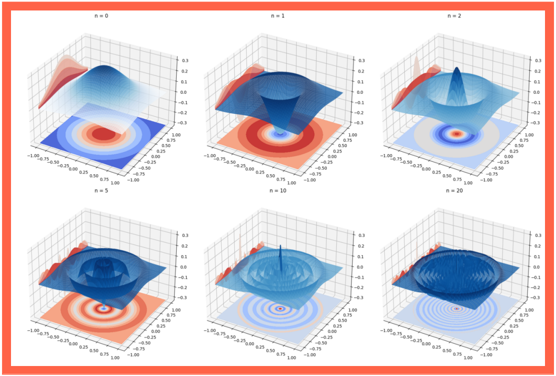
- Noncommutativity The bracket term measures the first way in which group multiplication differs from plain addition
- Structure Higher terms involve nested brackets, so the Lie algebra itself controls the whole expansion

Meaning over memorising



- Not the point The important thing is usually not the full series, but what it means conceptually
- Dictionary BCH translates a product in G into Lie-algebra language: sums plus correction brackets
- Local warning This is a near-the-identity statement; globally, the logarithm may fail or become multi-valued

Why we care: this is the bridge for later structure



- Use BCH explains how infinitesimal information remembers local multiplication in the group
- Consequence This is why Lie algebra maps are the right linear shadow of local Lie group maps
- Preview Next: homomorphisms can be differentiated to Lie algebra maps, and sometimes integrated back

Thank you for your attention!

Next time: The centre, derived algebra, and “simple” vs “solvable”