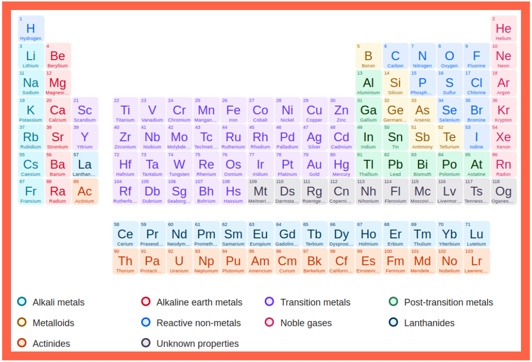
Lie theory - part 13

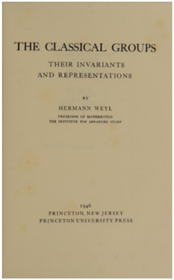
Or: Classification of simple Lie algebras in a nutshell

Why this classification matters



- **Elements of symmetry** Simple Lie algebras are the basic indivisible building blocks, so classifying them means finding the atoms of the story
- **Amazingly short list** Over $\mathbb{C}$, the answer is clean and memorable: four infinite families, plus five exceptional cases
- **Takeaway** This is one of those rare moments where a huge landscape collapses into a tidy cast list

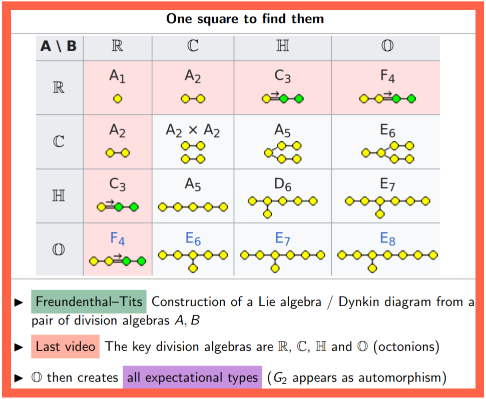
Most of the list comes in families



- A_n: $\mathfrak{sl}_{n+1}$, the special linear Lie algebra.
- B_n: $\mathfrak{so}_{2n+1}$, the odd-dimensional special orthogonal Lie algebra.
- C_n: $\mathfrak{sp}_{2n}$, the symplectic Lie algebra.
- D_n: $\mathfrak{so}_{2n}$, the even-dimensional special orthogonal Lie algebra ($n > 1$).

- **Type A** The family behind $\mathfrak{sl}_n$ is the first endless supply, and already captures a huge amount of the geometry and algebra we care about
- **Types B, C, D** Orthogonal and symplectic symmetries give the other classical families, so most simple Lie algebras come in these recurring patterns
- **In practice** A lot of Lie theory is learning how these four classical series behave

And then there are five exceptions



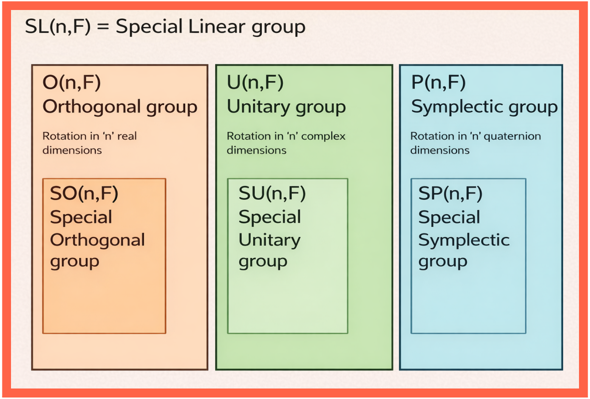
One square to find them

A \ B	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 \times A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

- Freundenthal–Tits Construction of a Lie algebra / Dynkin diagram from a pair of division algebras A, B
- Last video The key division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ (octonions)
- $\mathbb{O}$ then creates all expectational types (G_2 appears as automorphism)

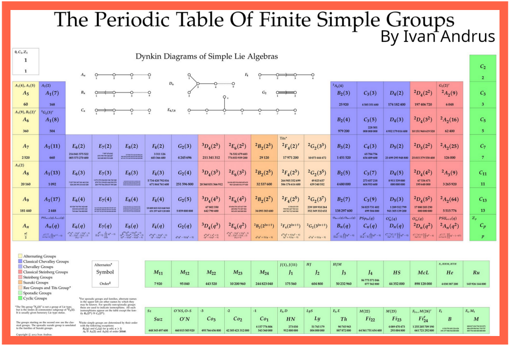
- The exceptional ones Beyond the classical families sit five isolated gems: G_2, F_4, E_6, E_7, E_8
- Why they matter They are rare, rigid, and a bit mysterious, which is exactly why people keep coming back to them
- The full classification is therefore “four infinite families plus five exceptions”

Where do these types come from?



- **Hidden geometry** After choosing the right commutative part, the Lie algebra breaks into patterns encoded by roots and their symmetries
- **Combinatorics wins** Those patterns can be packaged into Dynkin diagrams, turning a hard algebra problem into a crisp classification game
- **Outcome** "Continuous symmetries are rotations + epsilon"

A cousin: finite simple groups



- **Same spirit** In both stories, one wants to classify the simple building blocks from which more complicated symmetry objects are assembled
- **Very different difficulty** For simple Lie algebras, the answer is elegant and comparatively quick; for finite simple groups, the classification was vastly harder and took about a century longer to complete
- **However** The outcome is almost the same: continuous symmetries are like $SL_n(\mathbb{C})$, finite symmetries are like $SL_n(\mathbb{F}_p)$

Thank you for your attention!

Next time: Adjoint action