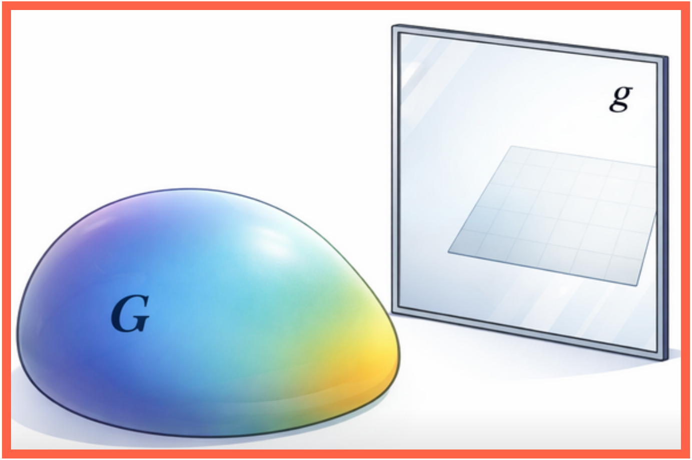
Lie theory - part 14

Or: Adjoint action: Ad and ad

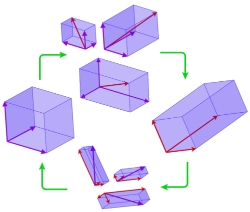
Why should a group act on its own Lie algebra?



- Global vs local Lie group = global object; Lie algebra = local linear shadow
- Conjugation lets the group move around its own infinitesimal directions
- The adjoint action is the group looking at itself through first-order geometry

Ad is conjugation made linear

Matrix problems



- A natural equivalence relation on matrices is similarity:

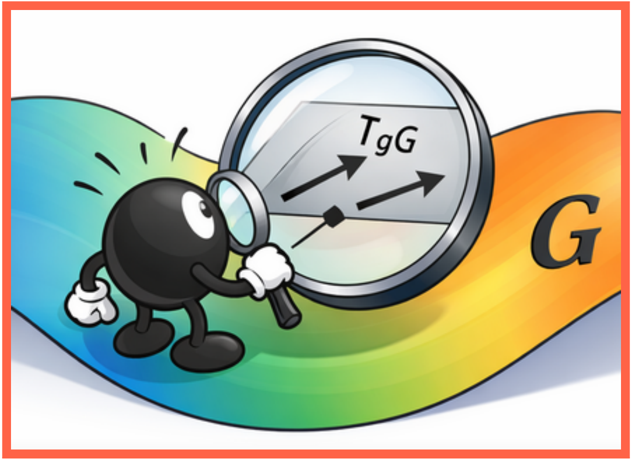
$$(A \sim B) \Leftrightarrow (\exists P : A = P^{-1}BP)$$

  Similarity = A and B are the same linear automorphism up to base change

- Question How can we classify similar matrices?

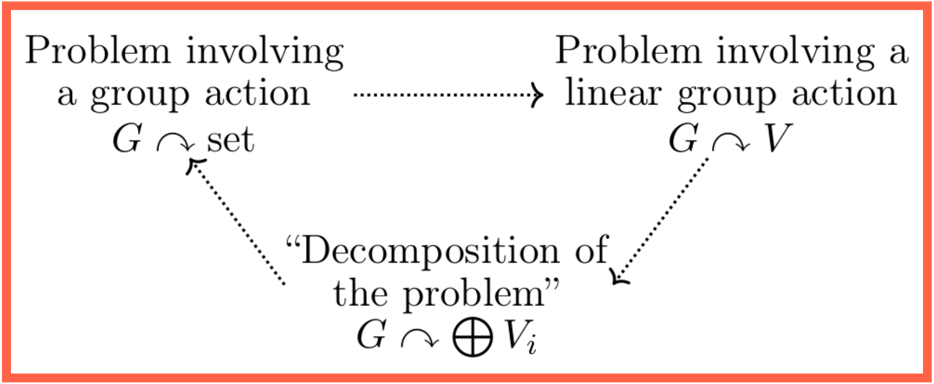
- Starting point Group elements conjugate other group elements
- Analogy Like similarity of matrices: different presentation, same structure
- Meaning of Ad Differentiate conjugation at the identity to get Ad

ad is the infinitesimal version



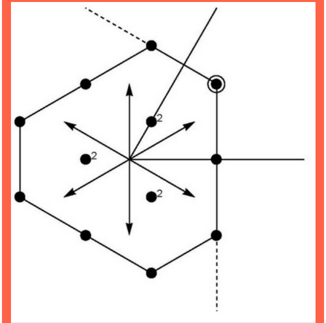
- Zoom in again ad is the Lie algebra version of Ad
- Analogy Think: map, derivative, then infinitesimal generator
- Matrix punchline For matrices, $\mathrm{ad}_X(Y) = [X, Y]$

Why ad is so useful?



- Center Central elements are exactly the ones with trivial ad-action
- Ideals Ideals are the subspaces stable under all ad_X
- Moral Internal structure becomes representation theory on the algebra itself

Matrix examples to keep in mind



Example of the weights of a representation of the Lie algebra sl(3,C)

- Concrete For matrix groups, Ad really is conjugation
- Diagonal case Diagonal matrices hint at weights and roots later on
- Rotation case In $SO(3)$, the adjoint action just says: if you rotate space, you also rotate the corresponding infinitesimal rotation axes

Thank you for your attention!

Next time: Representations of Lie groups: first principles