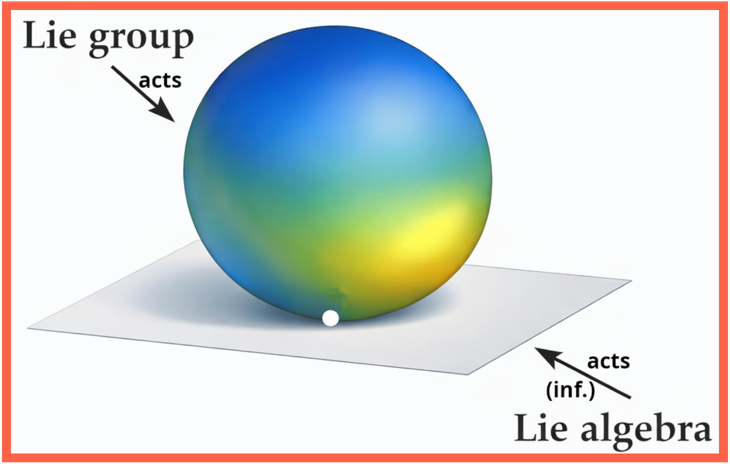
Lie theory - part 16

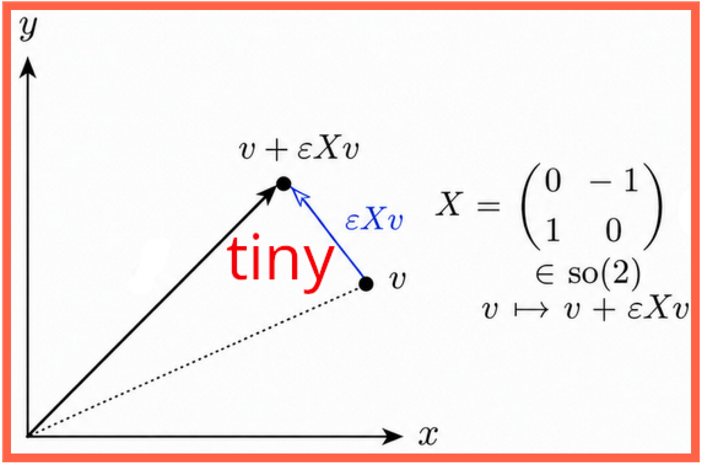
Or: Representations of Lie algebras

Why Lie algebra representations?



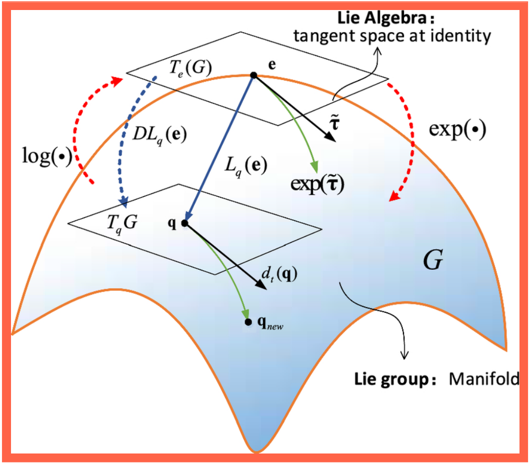
- Local symmetry Near id, a Lie group is controlled by tangent directions
- Linear motion A group action on V has infinitesimal operators on V
- Main idea Lie algebra reps are the derivatives of Lie group reps

What is the data?



- **Assignment** A vector $X \in \mathfrak{g}$ gives a linear map $\rho(X)\colon V \to V$
- **Compatibility** The bracket is respected: $\rho([X, Y]) = [\rho(X), \rho(Y)]$
- **Dictionary** The whole bracket is translated into commutators of matrices

From group reps to algebra reps



- **Start** Begin with a smooth group representation $\Pi\colon G \to \mathrm{GL}(V)$
- **Differentiate** Set $\rho(X) = \frac{d}{dt}\Big|_0 \Pi(\exp(tX))$
- **Meaning** Curves in G become first-order linear operators on V

Examples before definitions bite

Weyl ∼1923. The SL_2 (dual) Weyl modules $\Delta(v-1)$.

$\Delta(1-1)$: X^0Y^0

$\Delta(2-1)$: X^1Y^0 X^0Y^1

$\Delta(3-1)$: X^2Y^0 X^1Y^1 X^0Y^2

$\Delta(4-1)$: X^3Y^0 X^2Y^1 X^1Y^2 X^0Y^3

$\Delta(5-1)$: X^4Y^0 X^3Y^1 X^2Y^2 X^1Y^3 X^0Y^4

$\Delta(6-1)$: X^5Y^0 X^4Y^1 X^3Y^2 X^2Y^3 X^1Y^4 X^0Y^5

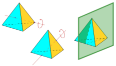
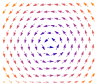
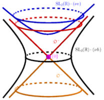
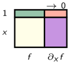
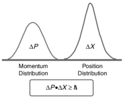
$\Delta(7-1)$: X^6Y^0 X^5Y^1 X^4Y^2 X^3Y^3 X^2Y^4 X^1Y^5 X^0Y^6

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto$ matrix who's rows are expansions of $(aX+cY)^{v-i}(bX+dY)^{i-1}$.

- Matrix action $\mathfrak{gl}_n$ acts on $\mathbb{C}^n$ by $X \cdot v = Xv$ (defining rep)
- Rotations $\mathfrak{so}_n$ gives infinitesimal rotations of Euclidean space
- The lab $\mathfrak{sl}_2$ is the toy model where much of the story can be seen

Big picture bridge

Abstract vs. real life

	Abstract	Incarnation
Numbers	3	 or  or...
Finite groups	$S_4 = \langle s, t, u \mid \text{some relations} \rangle$	 or  or...
Lie groups	$SL_2 = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \}$	 or  or...
More (Lie algebras, algebras,) categories...)	$W = \langle X, Y \mid XY = YX + 1 \rangle$	 or  or...

- **Easier world** Lie algebras are linear spaces, so their reps are often easier
- **Subtle** Not every infinitesimal story automatically integrates back to the group
- **Still useful** Lie algebra reps = local symmetry, even without a global group action

Thank you for your attention!

Next time: Simple vs reducible