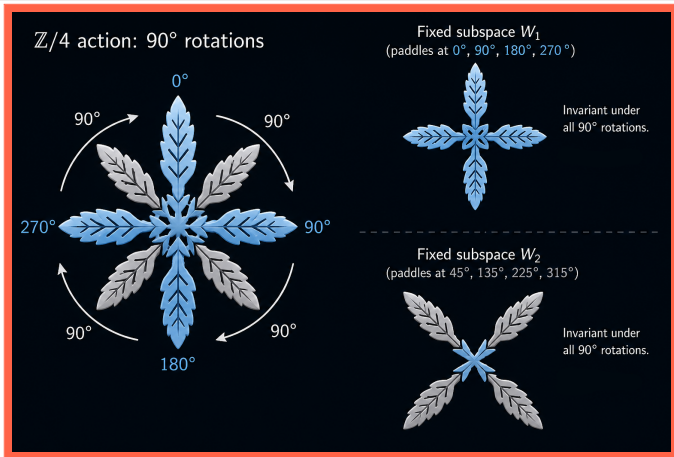


Lie theory - part 17

Or: Simple vs reducible

Why split a representation?



- ▶ **Idea** A complicated action may secretly be several simpler actions side by side
- ▶ **Invariant pieces** Some subspaces stay put under every symmetry in sight
- ▶ **Goal** Find the basic building blocks before asking for classification

Reducible vs irreducible

Simples are like elements

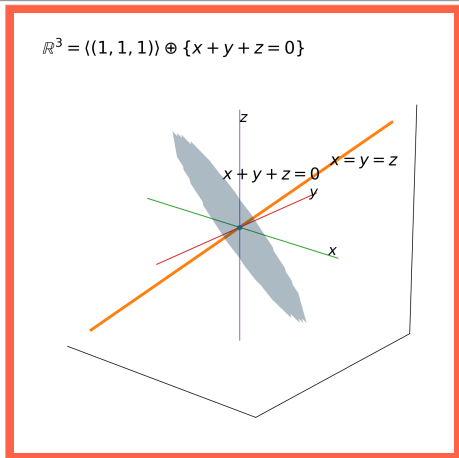
The periodic table is color-coded as follows:

- Alkali metals:** Blue circles (Li, Na, K, Rb, Cs, Fr)
- Alkaline earth metals:** Red circles (Be, Mg, Ca, Sr, Ba, Ra)
- Transition metals:** Purple circles (Ti, V, Cr, Mn, Fe, Co, Ni, Cu, Zn, Ga, Ge, As, Se, Br, Kr, Y, Zr, Nb, Mo, Tc, Ru, Rh, Pd, Ag, Cd, In, Sn, Sb, Te, I, Xe, Hf, Ta, W, Re, Os, Ir, Pt, Au, Hg, Tl, Pb, Bi, Po, At, Rn, Rf, Db, Sg, Bh, Hs, Mt, Ds, Rg, Cn, Nh, Fl, Mc, Lv, Ts, Og)
- Post-transition metals:** Green circles (Al, Ga, In, Tl, Sn, Pb, Bi, Po, At, Rn)
- Metalloids:** Yellow circles (B, Si, As, Sb, Te, Po, At, Rn)
- Reactive non-metals:** Light blue circles (C, N, O, F, Ne, P, S, Cl, Ar, Se, Br, Kr, Xe, I, Xe)
- Noble gases:** Pink circles (He, Ne, Ar, Kr, Xe, Rn)
- Lanthanides:** Light blue circles (La, Ce, Pr, Nd, Pm, Sm, Eu, Gd, Tb, Dy, Ho, Er, Tm, Yb, Lu)
- Actinides:** Orange circles (Ac, Th, Pa, U, Np, Pu, Am, Cm, Bk, Cf, Es, Fm, Md, No, Lr)
- Unknown properties:** Grey circles (Sc, Y, Zr, Nb, Mo, Tc, Ru, Rh, Pd, Ag, Cd, In, Sn, Sb, Te, I, Xe, Hf, Ta, W, Re, Os, Ir, Pt, Au, Hg, Tl, Pb, Bi, Po, At, Rn, Rf, Db, Sg, Bh, Hs, Mt, Ds, Rg, Cn, Nh, Fl, Mc, Lv, Ts, Og)

- ▶ **Reducible** Nontrivial invariant subspace \Rightarrow the action has an internal seam
- ▶ **Simple** No such seam \Rightarrow the representation behaves like one indivisible piece
- ▶ **Example** Diagonal matrices acting on \mathbb{C}^2 preserve the two coordinate lines

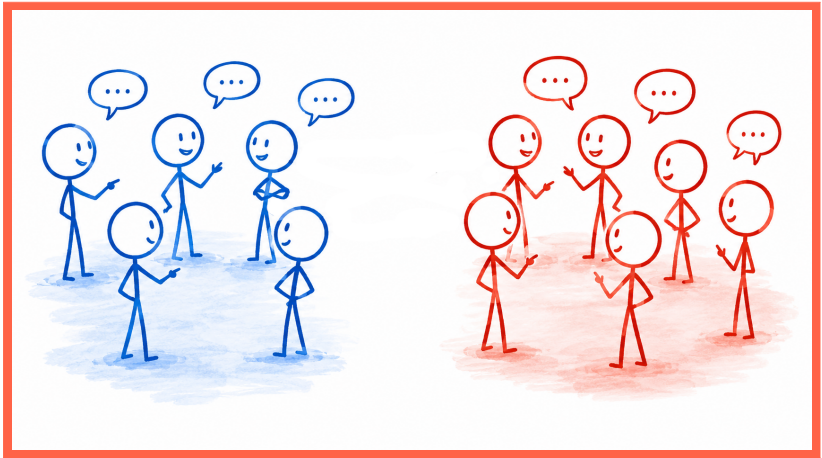
A favorite example: three coordinates

$$\mathbb{R}^3 = \langle (1, 1, 1) \rangle \oplus \{x + y + z = 0\}$$



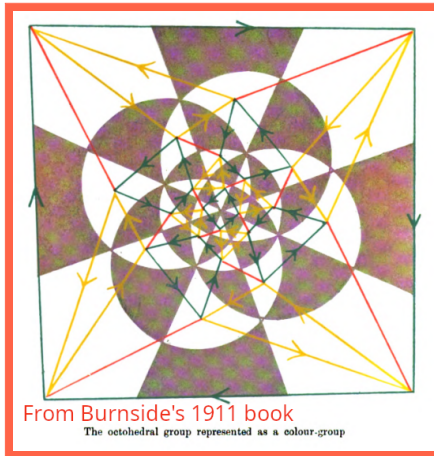
- ▶ Symmetry Permuting three coordinates acts on \mathbb{C}^3
- ▶ \oplus The all-equal line stays fixed, while the sum-zero plane moves around
- ▶ In formulas $\mathbb{C}^3 \cong \mathbb{C} \oplus \mathbb{C}^2$ as S_3 -reps

Schur's lemma, in human language



- ▶ **Rigidity** Maps between simples have remarkably little room to manoeuvre
- ▶ **Consequence** Different simples do not talk to each other
- ▶ **Why care** This turns allows us to compare, count, and eventually classify

Decomposition as a strategy



- ▶ **Toolkit** Break reps into pieces, understand the pieces, then rebuild the whole
- ▶ **Storytime** Averaging made this philosophy work beautifully for finite groups
- ▶ **Next bridge** Compact Lie groups inherit a similar miracle

Thank you for your attention!

Next time: Compact groups and complete reducibility