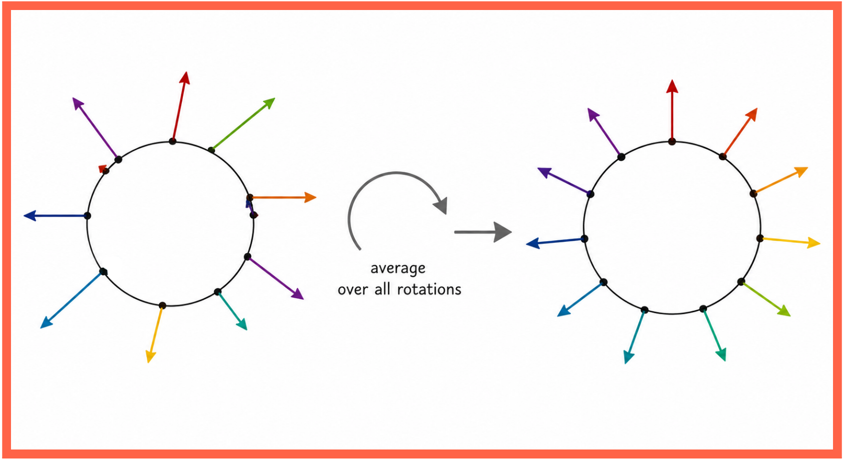
Lie theory - part 18

Or: Compact groups and complete reducibility

Compactness is averaging power

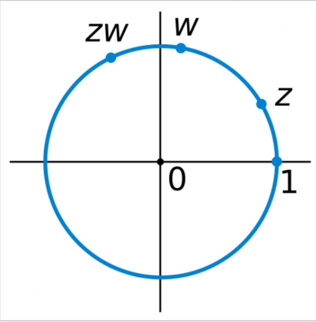


- Idea Finite symmetry can be averaged over; compact as well
- Effect Averaging turns arbitrary choices into symmetry-respecting choices
- Example Rotations preserve the circle's geometry: angles and lengths stay put

The unitary trick

In mathematics, a **compact (topological) group** is a topological group whose topology realizes it as a compact topological space. Compact groups are a natural generalization of finite groups with the discrete topology and have properties that carry over in significant fashion. Compact groups have a well-understood theory, in relation to group actions and representation theory.

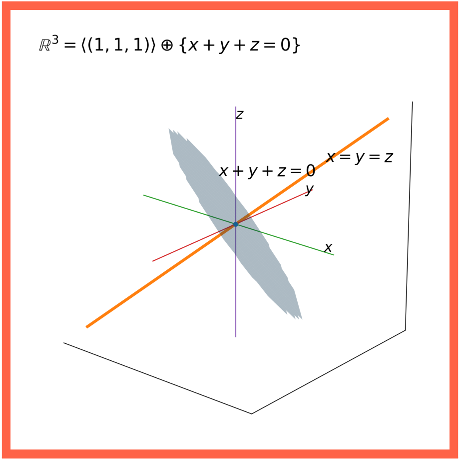
In the following we will assume all groups are Hausdorff spaces.



The circle of center 0 and radius 1 in the complex plane is a compact Lie group with complex multiplication.

- Start Put any inner product on the vector space, with no promises made
- Trick Average it around the group until every symmetry preserves it
- Storytime This is the friendly face of Weyl's "unitary trick"

How invariant subspaces split



- Setup Suppose a subspace survives every group action
- Orthogonal friend Its perpendicular complement survives as well
- Example $\mathbb{C}^3 \cong \mathbb{C} \oplus \mathbb{C}^2$ as S_3-reps

Complete reducibility

A simultaneous base change

S_3 acts on $\mathbb{C}[S_3]$:

$$(132) \rightsquigarrow \begin{array}{c|cccccc} & id & (12) & (23) & (123) & (132) & (13) \\ \hline id & 0 & 0 & 0 & 1 & 0 & 0 \\ (12) & 0 & 0 & 1 & 0 & 0 & 0 \\ (23) & 0 & 0 & 0 & 0 & 0 & 1 \\ (123) & 0 & 0 & 0 & 0 & 1 & 0 \\ (132) & 1 & 0 & 0 & 0 & 0 & 0 \\ (13) & 0 & 1 & 0 & 0 & 0 & 0 \end{array}$$

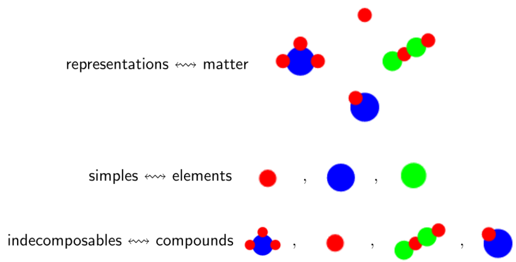
$$(132) \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- After base change, the matrices take block form
- This works simultaneously (above only one of six matrices)

- Repeat If a piece still has a seam, split it again
- Miracle For compact groups, this process gives a direct sum of simples
- Slogan Compactness prevents the usual Jordan-block misery

Why this changes the game

This is weird and surprising!



- Semisimplicity ⇔ "simple=indecomposable"
- Thus, for complex representations of finite groups we have

There are no nontrivial compounds!

- Reliable Decomposition is no longer wishful thinking, but a working machine
- Consequence Characters become useful because pieces really do separate
- Next bridge Peter–Weyl turns this into Fourier analysis with symmetry

Thank you for your attention!

Next time: Peter–Weyl in 10 minutes