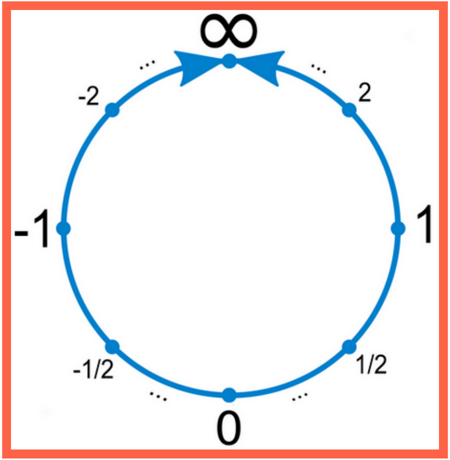
Lie theory - part 19

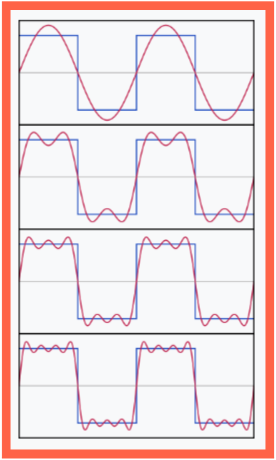
Or: Peter–Weyl in 10 minutes

Fourier analysis wants a new home



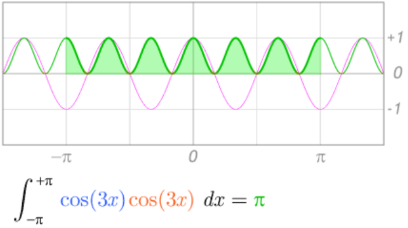
- **Classical** On $\mathbb{R}$, Fourier analysis uses the waves $e^{i\xi x}$ to decompose functions
- **Question** What replaces $e^{i\xi x}$ when the space is a compact group?
- **Example** For S^1, the waves are still $z \mapsto z^n$

The new waves are representations



- Dictionary A representation $\rho\colon G \to GL(V)$ gives functions on G
- Name Matrix coefficients look like $g \mapsto \langle v^*, \rho(g)v \rangle$
- Example For S^1, the matrix coefficients are just the usual waves $z \mapsto z^n$

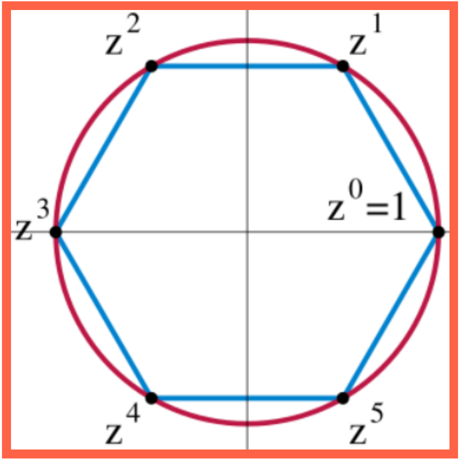
Orthogonality is the sorting machine



Sines and cosines form an orthogonal set, as illustrated above. The integral of sine, cosine and their product is zero (green and red areas are equal, and cancel out) when m, n or the functions are different, and π only if m and n are equal, and the function used is the same. They would form an orthonormal set, if the integral equaled 1 (that is, each function would need to be scaled by $1/\sqrt{\pi}$).

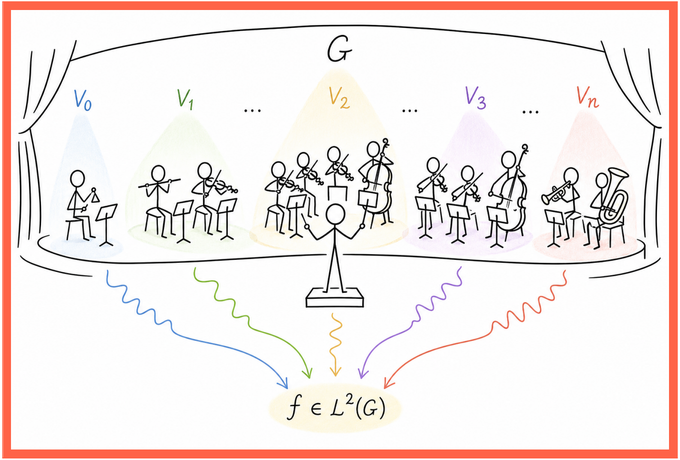
- **Memory** Last time: compact representations split into simple pieces
- **Consequence** Different simples are orthogonal under $\int_G f(g)\overline{h(g)}\, dg$
- **Example** On S^1, this says $\int_0^{2\pi} e^{in\xi}\overline{e^{im\xi}}\, d\xi = 0$ for $n \neq m$

The punchline (Peter–Weyl theorem)



- Theorem Functions on G have Fourier series built from simple reps
- Upshot $L^2(G)$ is built from all simple reps (via matrix coeff.)
- Example On S^1, the simple reps are $z \mapsto z^n$ so we recover Fourier

Example: $SU(2)$



- **Simples** $SU(2)$ has simples $V_0, V_1, V_2, \ldots$, one in each dimension $1, 2, 3, \ldots$
- **Waves** Each V_n contributes matrix coefficients $g \mapsto \langle v^*, gv \rangle$
- **Fourier** Peter–Weyl says these form the Fourier blocks for $L^2(SU(2))$

Thank you for your attention!

Next time: Characters as fingerprints