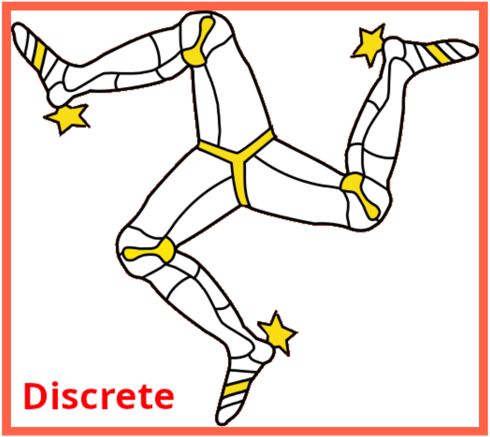
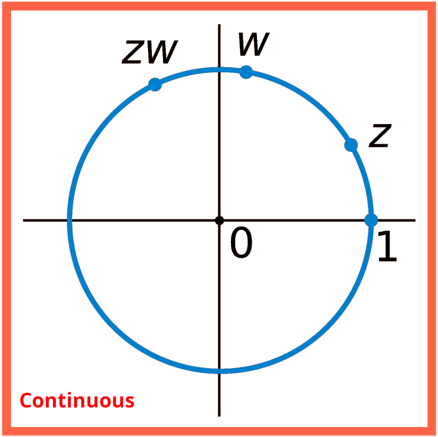
Lie theory - part 2?

Or: Where do Lie groups come from?

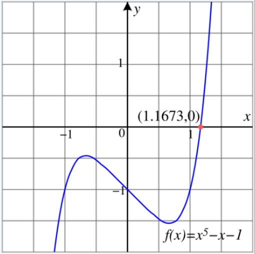
Origins!





- Lie theory = the mathematics of continuous/smooth symmetry
- Origins Polynomials $\rightsquigarrow$ finite symmetry, differential equations $\rightsquigarrow$ smooth symmetry
- Today's slogan "Symmetry of equations $\Rightarrow$ a group" (discrete vs. continuous)

Origin story I: symmetries of polynomial equations

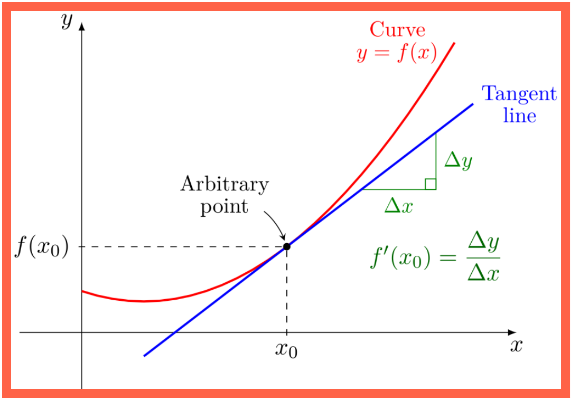


For the polynomial $f(x) = x^5 - x - 1$, the lone real root $x = 1.1673\ldots$ is algebraic, but not expressible in terms of radicals. The other four roots are complex numbers.

Symmetry group is S_5

- Polynomial equation $f(x) = 0$ has finitely many roots
- Symmetry Permute the roots without changing any algebraic relations over the base field $\Rightarrow$ a subgroup of the symmetric group S_n
- Galois group $=$ finite group capturing the root symmetries (solvability controlled "by radicals")

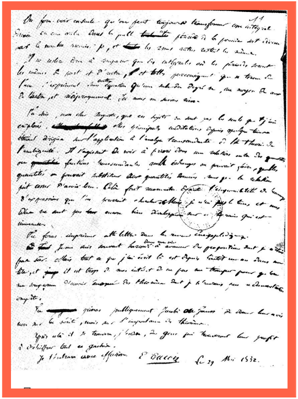
Why discrete?



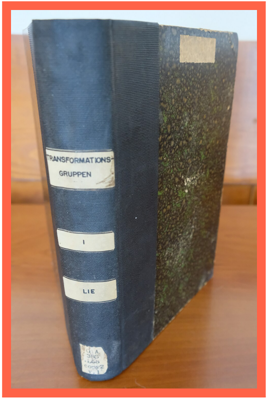
- Polynomials have finitely many roots
- Typical outcome The symmetry group is finite (often even S_n): it is about combinatorics of roots, not smooth motion
- Enter: geometry and calculus For "continuous symmetry" you need a setting where transformations can vary smoothly

Origin story II: symmetries of differential equations (DE)

Galois:

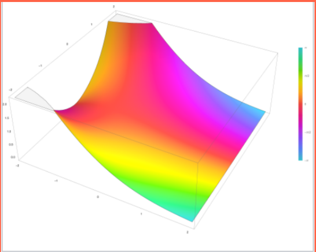


Lie:

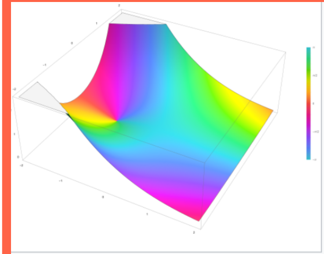


- A transformation g is a **symmetry of a DE** if it sends solutions to solutions
- **Continuous families** If symmetries depend smoothly on parameters (translations, rotations, scalings, ...), they form a Lie group
- **This is where they originate** Lie (1870s) wanted a continuous version of Galois' theory (1830s)

Example: a DE with a (continuous) symmetry group



Plot of the Airy function Ai(z) in the complex plane from -2 - 2i to 2 + 2i with colors created with Mathematica 13.1 function ComplexPlot3D



Plot of the derivative of the Airy function Ai'(z) in the complex plane from -2 - 2i to 2 + 2i with colors created with Mathematica 13.1 function ComplexPlot3D

- Airy equation (1838) $y'' - x\,y = 0$ (two famous solutions: $\mathrm{Ai}(x), \mathrm{Bi}(x)$.)
- Differential Galois group The "continuous Galois group" over $\mathbb{C}(x)$ is $SL_2(\mathbb{C})$
- Consequence Since $SL_2(\mathbb{C})$ is not solvable, the equation has no "elementary" solution: you must leave the zoo and use special functions

Thank you for your attention!

I hope that was of some help.