Lie theory - part 20

Or: Characters as fingerprints

Why remember traces?

My wish list

To each rep I want an associated numerical invariant

Numerical invariant = something like a number

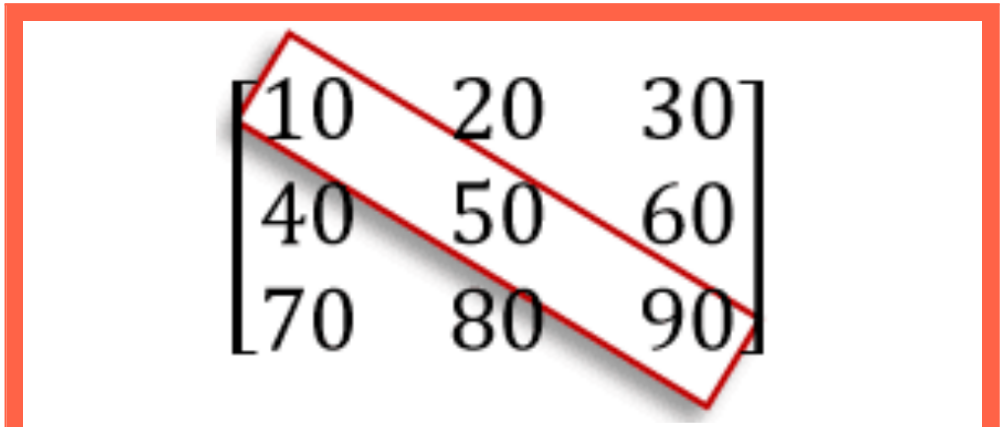
The invariant should behave nicely wrt operations on reps

The invariant should determine the rep

- The idea of invariants is ubiquitous in mathematics/the sciences
- So let's apply it in rep theory!
- However, the last point sounds impossible

- **Recall** A representation $\rho\colon G \to GL(V)$ assigns a matrix to each $g \in G$
- **Character** Its character is the trace function $\chi_\rho(g) = tr(\rho(g))$
- **Surprise** For compact groups, this small shadow remembers a lot

Characters ignore coordinates



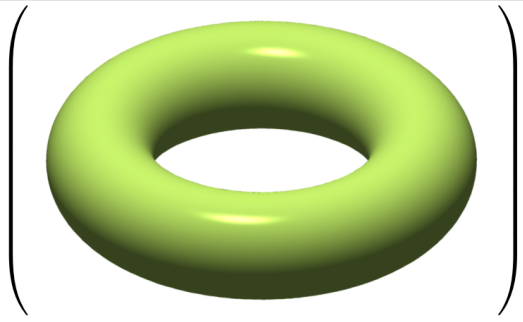
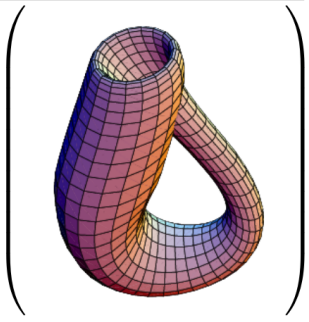
$$\operatorname{tr}(\mathbf{A}+\mathbf{B}) = \operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B})$$
$$\operatorname{tr}(c\mathbf{A}) = c\operatorname{tr}(\mathbf{A})$$
$$\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$$

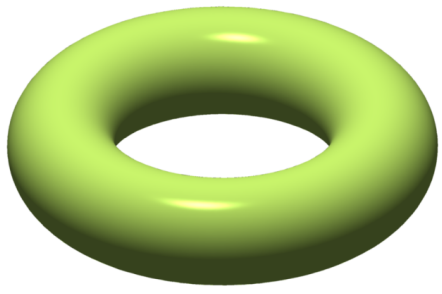
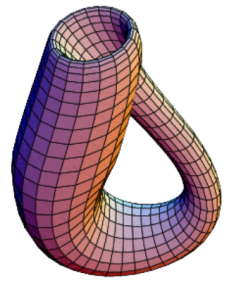
$$\operatorname{tr}(\mathbf{A}\otimes\mathbf{B}) = \operatorname{tr}(\mathbf{A})\operatorname{tr}(\mathbf{B})$$

- Basis Changing basis replaces a matrix by a conjugate matrix
- Trace But traces do not change: $tr(ABA^{-1}) = tr(B)$
- Class function So $\chi_\rho(hgh^{-1}) = \chi_\rho(g)$

Fingerprints of representations

The finite group miracle

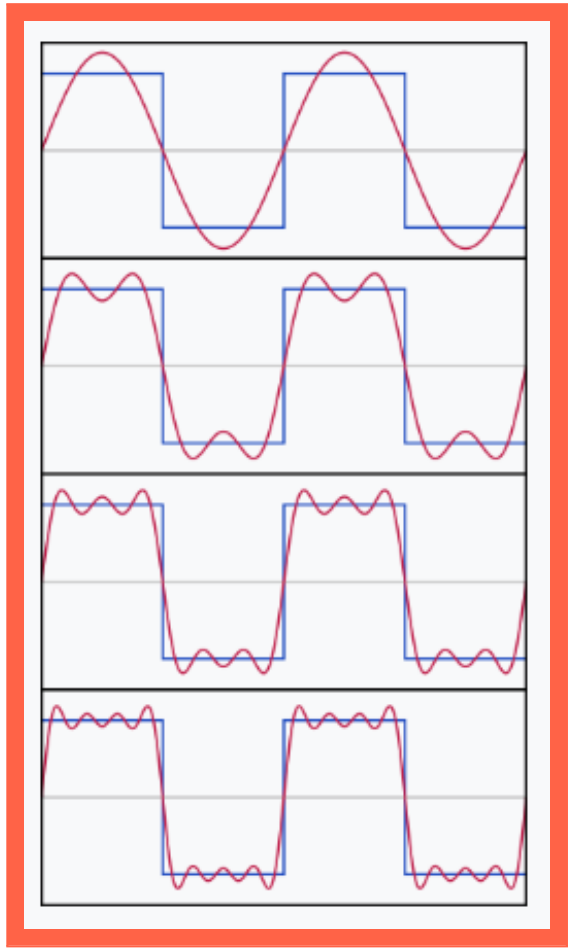
Euler char  = Euler char 

 $\not\cong$ 

- Over $\mathbb{C}$: $\phi = \psi \Leftrightarrow \chi_\phi = \chi_\psi$ Character determines rep
- Analogs in other fields are often very wrong

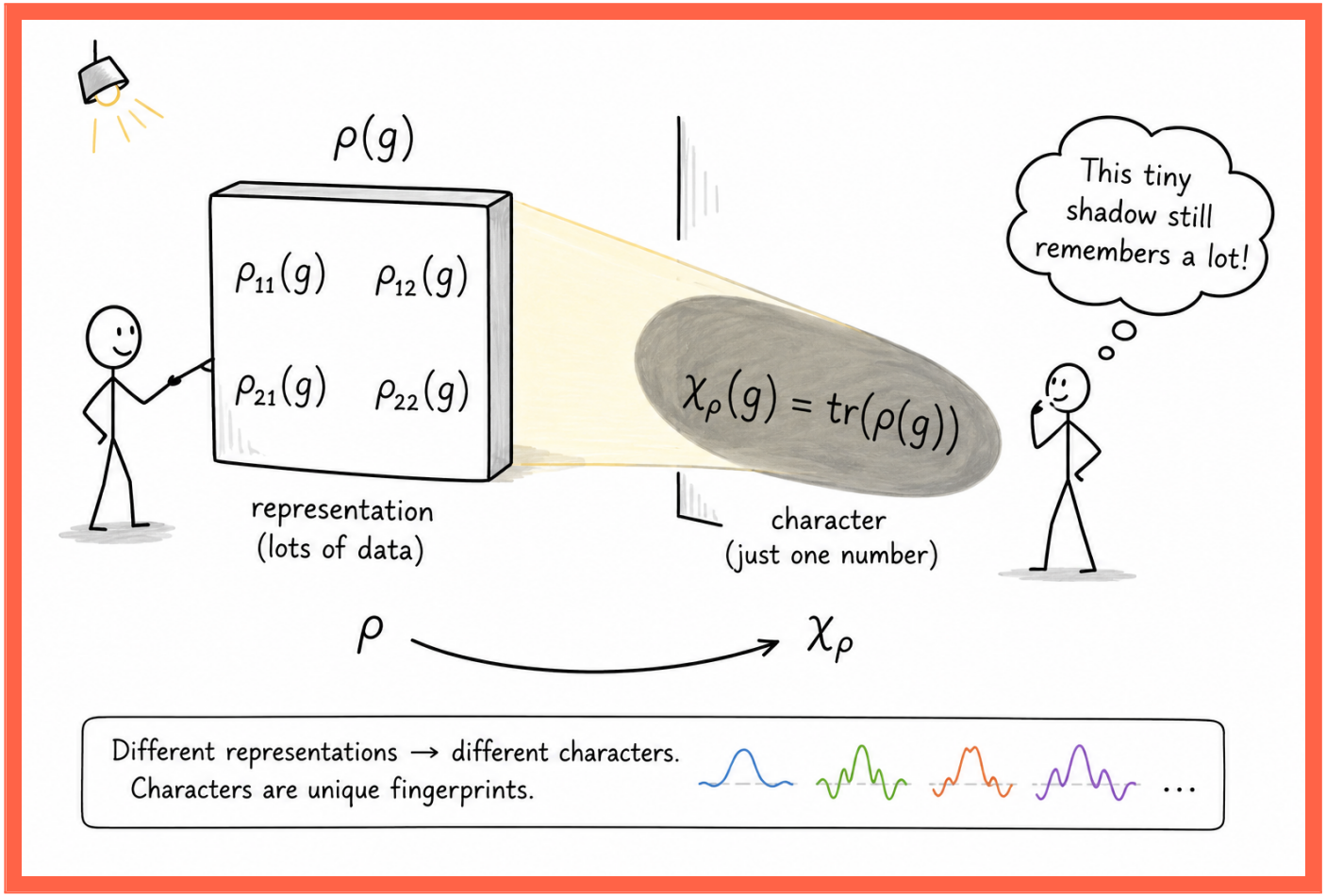
- Polynomial $\chi_{V\oplus W} = \chi_V + \chi_W$ and $\chi_{V\otimes W} = \chi_V \chi_W$
- Orthogonality Simple characters behave like perpendicular fingerprints
- Consequence Inner products with characters detect which simples occur

Example: the circle again



- **Simples** For S^1, the irreducibles are $z \mapsto z^n$ with $n \in \mathbb{Z}$
- **Characters** The characters are the usual Fourier modes $\chi_n(z) = z^n$
- **Orthogonality** The formula $\int_{S^1} z^n \overline{z^m}\, dz = \delta_{n,m}$ is the familiar story

Example: a first glimpse of $SU(2)$



- **Simples** The simples of $SU(2)$ are $V_0, V_1, V_2, \ldots$
- **Character** On $diag(e^{i\theta}, e^{-i\theta})$, one gets $\chi_n = \frac{\sin((n+1)\theta)}{\sin(\theta)}$
- **Moral** A representation can be recognized from a single trace function

Thank you for your attention!

Next time: The lab: $SU(2)$ and $\mathfrak{su}(2)$