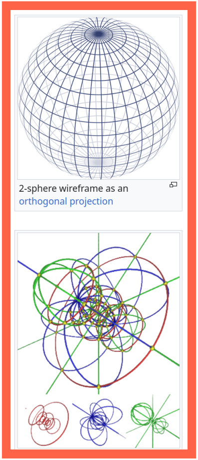
Lie theory - part 21

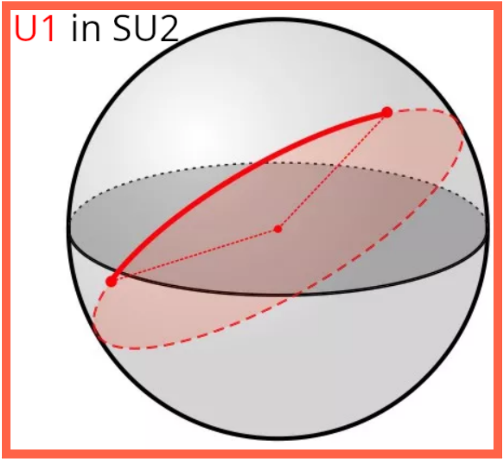
Or: The lab: $SU(2)$ and $\mathfrak{su}(2)$

Now actually meet $SU(2)$



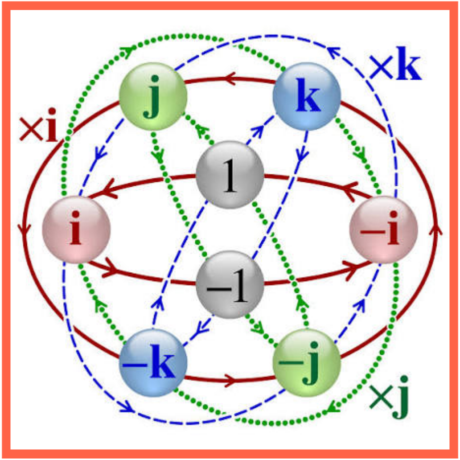
- Definition $SU(2)$: unitary 2×2 complex matrices with determinant 1
- Familiar face $diag(e^{i\theta}, e^{-i\theta})$ is our matrix from last time
- Laboratory Compact, three-dimensional, noncommutative $\Rightarrow$ small but rich

Every element looks like this



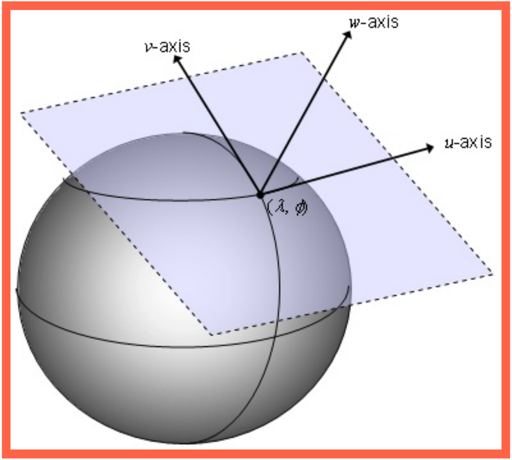
- Concrete Every element is $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$
- Special cases $b = 0$ gives the circle $U(1)$; real a, b give rotations
- Payoff Two complex numbers + one equation = three real parameters

A 3-sphere hiding in matrices



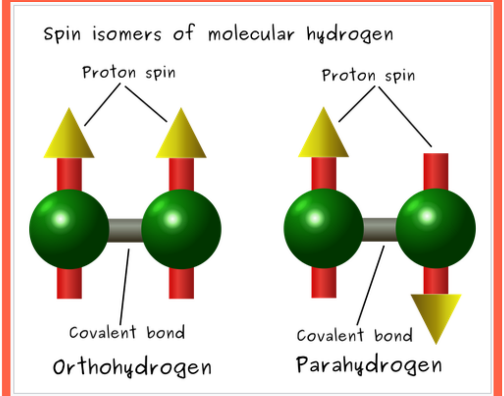
- Coordinates The pair $(a, b) \in \mathbb{C}^2$ is just four real coordinates
- Geometry $|a|^2 + |b|^2 = 1$ cuts out the unit 3-sphere $S^3 \subset \mathbb{R}^4$
- Quaternion model Unit quaternions are another model of the same group

Zoom in: $\mathfrak{su}(2)$



- Tangent $\mathfrak{su}(2) = T_I SU(2)$ consists of traceless skew-Hermitian matrices
- Concrete Every X is $\begin{pmatrix} ir & z \\ -\bar{z} & -ir \end{pmatrix}$ with $r \in \mathbb{R}$ and $z \in \mathbb{C}$
- Size Again there are three real directions: the infinitesimal shadow of $SU(2)$

Three directions, one bracket



Spin isomers of molecular hydrogen

- Spin Multiplying the Pauli matrices by $-i/2$ puts them inside $\mathfrak{su}(2)$
- Commutator For $J_k = -\frac{i}{2}\sigma_k$, one has $[J_x, J_y] = J_z$ and cyclically
- Moral Geometry, brackets and quantum spin all meet in this tiny laboratory

Thank you for your attention!

Next time: Classifying irreps of $SU(2)$