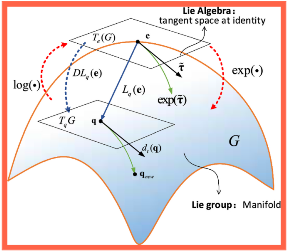
Lie theory - part 22

Or: Classifying irreps of $SU(2)$

The answer in one line

Back to exponentials



- Direction → motion If $X \in \mathfrak{g}$, then $t \mapsto e^{tX}$ stays inside G
- Interpretation The exponential turns an "allowed infinitesimal move" into a genuine one-parameter subgroup
- Next step Now we can ask: what extra structure lives on $\mathfrak{g}$? (that's where the bracket enters)

- Classification Every simple is $V_n = \mathrm{Sym}^n(\mathbb{C}^2)$ for $n = 0, 1, 2, \ldots$
- Dimensions $\dim V_n = n + 1$: exactly one simple in every positive dimension
- Plan Find a top weight, then let the $\mathfrak{sl}_2 = \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}$ ladder do the work

What are those $x^i y^j$?

Weyl ∼1923. The SL_2 Weyl modules $\Delta(v-1)$.

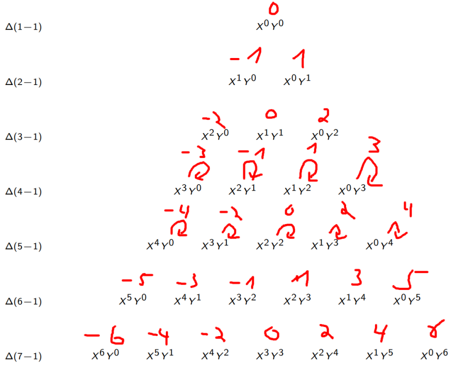
$$
\begin{array}{lccccccccccccc}
\Delta(1-1) & & & & & & & X^0Y^0 & & & & & & \\
\Delta(2-1) & & & & & & X^1Y^0 & & X^0Y^1 & & & & & \\
\Delta(3-1) & & & & & X^2Y^0 & & X^1Y^1 & & X^0Y^2 & & & & \\
\Delta(4-1) & & & & X^3Y^0 & & X^2Y^1 & & X^1Y^2 & & X^0Y^3 & & & \\
\Delta(5-1) & & & X^4Y^0 & & X^3Y^1 & & X^2Y^2 & & X^1Y^3 & & X^0Y^4 & & \\
\Delta(6-1) & & X^5Y^0 & & X^4Y^1 & & X^3Y^2 & & X^2Y^3 & & X^1Y^4 & & X^0Y^5 & \\
\Delta(7-1) & X^6Y^0 & & X^5Y^1 & & X^4Y^2 & & X^3Y^3 & & X^2Y^4 & & X^1Y^5 & & X^0Y^6
\end{array}
$$

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto$ matrix who's columns are expansions of $(aX+cY)^{v-i}(bX+dY)^{i-1}$.

- **Model** Choose a basis x, y of $\mathbb{C}^2$; $V_n = \mathrm{span}\{x^n, x^{n-1}y, \dots, xy^{n-1}, y^n\}$
- **Meaning** $x^i y^j$ means i copies of x and j copies of y, symmetrized; $i+j=n$
- **Example** In V_2: $ax^2 + bxy + cy^2$

Let $SU(2)$ act on the variables

Weyl ~1923. The SL_2 Weyl modules $\Delta(v-1)$.

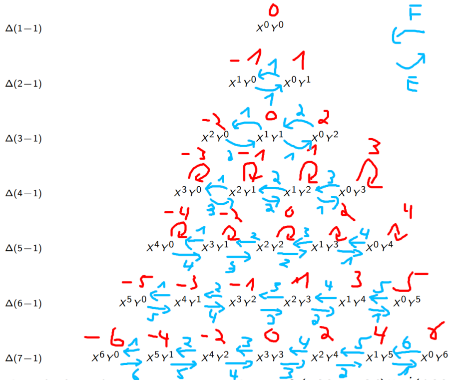


$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$ matrix who's columns are expansions of $(aX + cY)^{v-i}(bX + dY)^{i-1}$.

- **Diagonal** $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$ sends $x \mapsto e^{i\theta}x$ and $y \mapsto e^{-i\theta}y$
- **Monomials** Hence $x^{n-k}y^k \mapsto e^{i(n-2k)\theta}x^{n-k}y^k$
- **Read it off** Weights $n, n-2, \ldots, -n$ are read straight from the monomials

The ladder becomes concrete

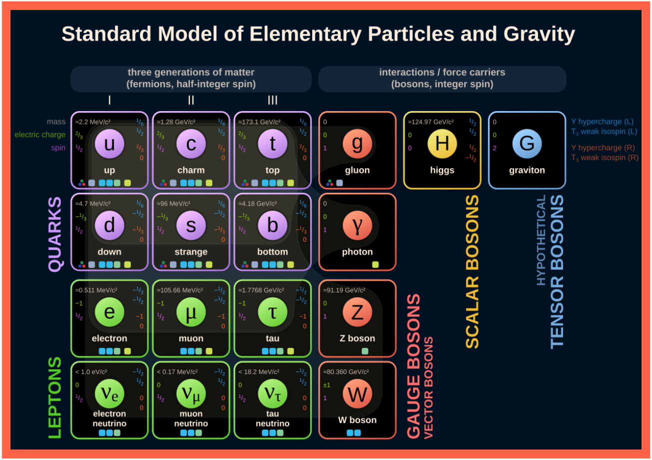
Weyl ~1923. The SL_2 Weyl modules $\Delta(v-1)$.



$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$ matrix who's columns are expansions of $(aX + cY)^{v-i}(bX + dY)^{i-1}$.

- ▶ Operators In $\mathfrak{sl}_2$: $E = x\partial_y$ raises, $F = y\partial_x$ lowers
- ▶ Top rung $E(x^n) = 0$; applying F turns one x into a y
- ▶ Classification Every simple is one such finite ladder

Physicists call this spin



- **Particle language** A spin-j particle has $2j+1$ possible spin states
- **Spin $\frac{1}{2}$** An electron has two states: $|\uparrow\rangle$ and $|\downarrow\rangle$
- **Connection** These states form the $(2j+1)$-dim simple rep of $SU(2)$

Thank you for your attention!

Next time: Tensor products for $SU(2)$