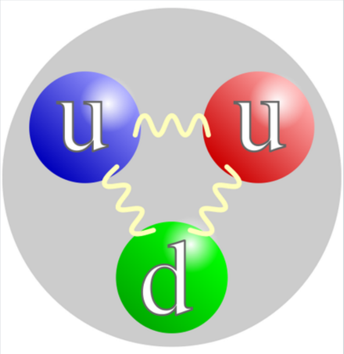
Lie theory - part 23

Or: Tensor products for $SU(2)$

One spin is only the beginning



Proton quark structure: 2 up quarks and 1 down quark.

- One electron Spin $\frac{1}{2}$: a state in $V_1 = \mathbb{C}^2$, with basis $|\uparrow\rangle, |\downarrow\rangle$
- Real physics Spin-0 electron pairs underpin conventional superconductivity
- Physical question What total spin can we measure for the pair?

Why a tensor product?

- Combined space $V_1 \otimes V_1 = \mathbb{C}^2 \otimes \mathbb{C}^2$: four dimensions
- Basis $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$ record both spins
- Quantum twist Superpositions include entangled states of the pair

Rotate the pair together

$$\left(\mathbb{R}^2 \xrightarrow{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \mathbb{R}^2 \right) \otimes \left(\mathbb{R}^3 \xrightarrow{\begin{pmatrix} 5 & 6 & 7 \end{pmatrix}} \mathbb{R}^1 \right)$$

$$= \mathbb{R}^6 \xrightarrow{\left(\begin{array}{cc|cc|cc} 5 & 10 & 6 & 12 & 7 & 14 \\ 15 & 29 & 18 & 24 & 21 & 28 \end{array}\right)} \mathbb{R}^2$$

They are intertwined in a multiplicative way.

- **Same rotation** $g(v \otimes w) = gv \otimes gw$: rotate both spins
- **Careful** Opposite arrows need not mean total spin zero
- **Our task** Find the total-spin sectors: spin J corresponds to V_{2J}

Two spin halves: triplet and singlet

- **The split** $V_1 \otimes V_1 \cong V_2 \oplus V_0$: $2 \cdot 2 = 3 + 1$
- **Triplet** Spin 1: the three-dimensional symmetric subspace
- **Singlet** Spin 0: $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$

The Clebsch–Gordan rule

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- General rule $V_m \otimes V_n \cong \bigoplus_{k=0}^{\min(m,n)} V_{m+n-2k}$
- Example $V_1 \otimes V_2 \cong V_3 \oplus V_1$: spin $\frac{1}{2}$ and 1 give $\frac{3}{2}$ or $\frac{1}{2}$
- Total spins From $|j_1 - j_2|$ to $j_1 + j_2$, in steps of 1, each once

Thank you for your attention!

Next time: From $SU(2)$ to $SO(3)$: the double cover