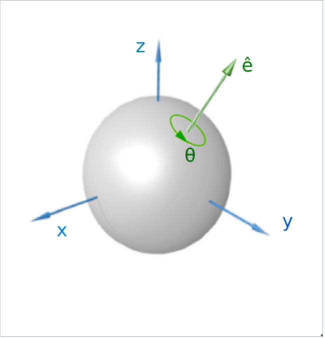
Lie theory - part 24

Or: From $SU(2)$ to $SO(3)$: the double cover

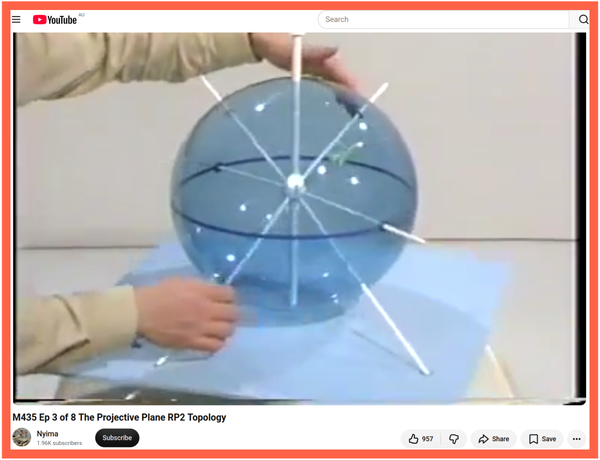
Why two rotation groups?



A visualization of a rotation represented by an Euler axis and angle.

- Ordinary space $SO(3)$ rotates vectors in $\mathbb{R}^3$
- Electron spin $SU(2)$ rotates vectors ("spinors") in $\mathbb{C}^2$
- The puzzle Both describe rotations, so why are they different groups?

Two lifts, one rotation



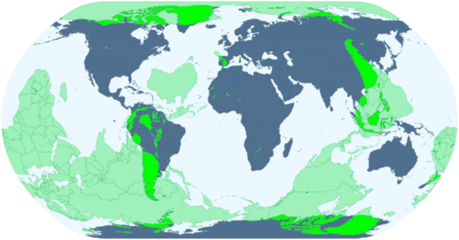
- The map Conjugation on $\mathfrak{su}(2) \cong \mathbb{R}^3$ gives $SU(2) \to SO(3)$
- Double cover Every rotation has two lifts: $\pm g$ and $SO(3) \cong SU(2)/\{\pm I\}$
- Topology $SO(3) \cong \mathbb{RP}^3$ and $SU(2) \cong S^3$

One turn gives a minus sign

Enter, the theorem

$SO_3(\mathbb{R})$ is not simply connected and its π_1 is $\mathbb{Z}/2\mathbb{Z}$

- $SO_3(\mathbb{R})$ = rotation group on $\mathbb{R}^3$
- Topologically $SO_3(\mathbb{R}) = S^3$/antipodal points



- Belt trick = a loop by 360° is nontrivial, doing it twice is trivial

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- Follow a turn A full 360° turn lifts from I to $-I$ in $SU(2)$
- Spin half The spinor changes as $\psi \mapsto -\psi$; 720° gives ψ again
- Same locally $\mathfrak{su}(2) \cong \mathfrak{so}(3)$; the difference is global

Which representations reach $SO(3)$?

Bosons [edit]

Bosons are one of the two fundamental particles having integral spinclasses of particles, the other being fermions. Bos characterized by Bose–Einstein statistics and all have integer spins. Bosons may be either elementary, like photons anc gluons, or composite, like mesons.

According to the Standard Model, the elementary bosons are:

Name	Symbol	Antiparticle	Spin [$\hbar$]	Charge [e]	Mass[9] [GeV/c^2]	Interaction mediated	Observ
photon	γ	self	1	0	0	electromagnetism	yes
W boson	W^-	W^+	1	±1	80.385 ± 0.015	weak interaction	yes
Z boson	Z	self	1	0	91.1875 ± 0.0021	weak interaction	yes
gluon	g	self	1	0	0	strong interaction	yes
Higgs boson	H^0	self	0	0	125.09 ± 0.24	mass	yes

- **Descent test** g and $-g$ must act identically: $\rho(-I) = I$
- **On V_n** $-I$ acts by $(-1)^n$, so exactly the even n pass
- **Integer spins (bosons)** $SO(3)$ simples: V_0 (trivial), V_2 (defining), $V_4, \ldots$

Can we see that minus sign?

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Spin Precession During Interferometry of Fermions and the Phase Factor Associated with Rotations Through 2π Radians

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Herbert J. Bernstein*

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Observability of the Sign Change of Spinors under 2π Rotations

Yakir Aharonov and Leonard Susskind

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Phys. Rev. **158**, 1237 – **Published 25 June, 1967**

DOI: https://doi.org/10.1103/PhysRev.158.1237

- **On its own** ψ and $-\psi$ give the same measurement probabilities
- **Interference** A relative minus sign between two paths changes the fringes
- **Two electrons** The signs cancel: $(-\psi) \otimes (-\phi) = \psi \otimes \phi$

Thank you for your attention!

Next time: Maximal tori and the diagonalisation philosophy