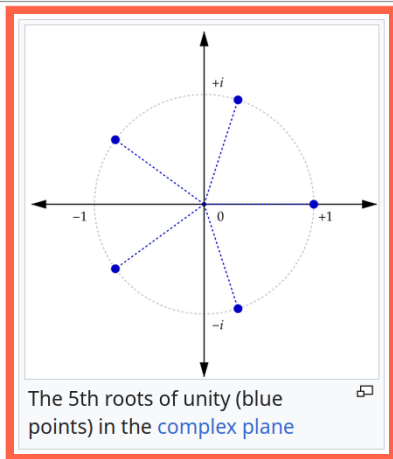


Lie theory - part 29

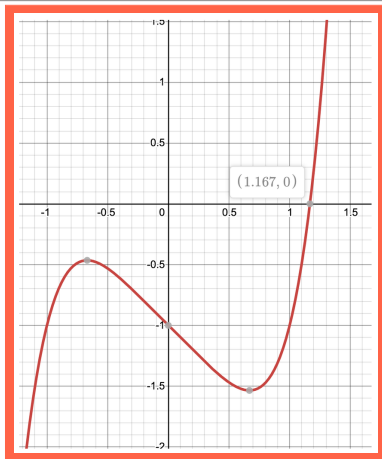
Or: Differential equations and symmetry

Galois theory: symmetries of roots



- **Roots** $X^5 - 1 = 0$: $1, \zeta, \zeta^2, \zeta^3, \zeta^4$, with $\zeta = e^{2\pi i/5}$
- **Rules** Permute roots while fixing $\mathbb{Q}$ and preserving sums and products
- **Group** $\zeta \mapsto \zeta^a$ for $a = 1, 2, 3, 4$: $G \cong C_4$

Same degree, different symmetry



- **Group** $X^5 - X - 1$ has $G = S_5$: all permutations of its five roots
- **Test** Radical formulas exist exactly when the Galois group is solvable
- **Verdict** C_4 solvable, S_5 nonsolvable: radicals work only for $X^5 - 1$

Differential Galois theory: matrix symmetries

Why is differential Galois theory not widely used?

Asked 11 years, 6 months ago Modified 1 year ago Viewed 23k times



E.R. Kolchin has developed the differential Galois theory in 1950s. And it seems powerful a tool which can decide the solvability and the form of solutions to a given differential equation.(e.g. Whether it is of closed form or not. see)

93



My question is why differential Galois theory is not widely used in differential geometry. It is plausible that we can solve some problems of differential/integral geometry using this set of theory.



I have read some answers provided here like [Why do we need admissible isomorphisms for differential Galois theory?](#) and other stuffs. I have read Kaplansky's and Bulum's books. My question follows:

So what is the major 'pullback' in this theory that prevents its wide application to other situations rather than discrete geometry (e.g. Diophantine geometry)?

My original question on Mathematics Stack Exchange is: [Why differential Galois theory is not widely used?](#) which yields no satisfying answers.

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edited Mar 20, 2022 at 4:57



Semion98
3 5

asked Apr 3, 2015 at 2:18

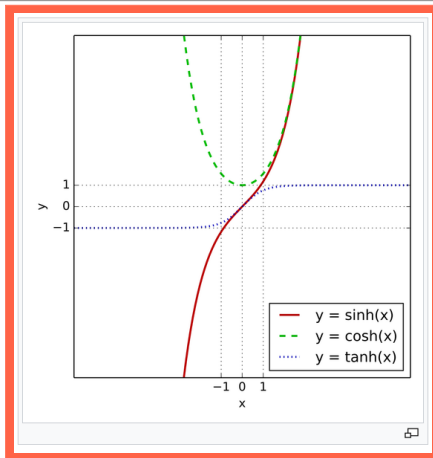


Henry L.
8,334 9 52 77

57 Differential Galois theory is concerned with linear differential equations in one variable. Specialists in differential equations tend to care about more than that (e.g., PDE, nonlinear equations), and the aspect of solvability addressed by differential Galois theory is largely of no interest to them even for linear differential equations. Even ordinary Galois theory is not really of much importance to analysts needing roots of a polynomial in $\mathbb{C}$. Nobody really cares about being able write roots in a very restrictive form related to solvable Galois groups. It is mostly of historic interest. – KConrad Apr 3, 2015 at 2:54 ✓

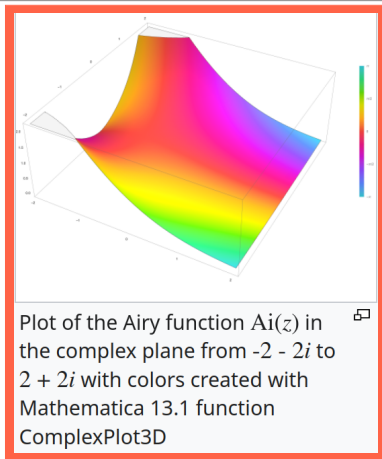
- **Space** Two independent solutions of $y'' = r(z)y$ form a basis
- **Group** Fix rational functions, respect algebra and derivatives: $G \subseteq GL_2(\mathbb{C})$
- **Test** $\text{Lie}(G)$ solvable iff algebraic functions, exponentials and integrals suffice

A solvable matrix group



- Equation $y'' = y$ has basis e^z, e^{-z}
- Constraint $e^z e^{-z} = 1$ forces the two scaling factors to be inverse
- Group $G = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} : c \in \mathbb{C}^\times \right\}$ is abelian, hence solvable

A nonsolvable Lie algebra obstructs formulas



- **Airy** $y'' = zy$ is the Airy equation
- **Group** $G = SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$
- **Verdict** $\mathfrak{sl}_2(\mathbb{C})$ is not solvable: no nonzero solution by the allowed operations

Thank you for your attention!

Next time: Fourier analysis with symmetry