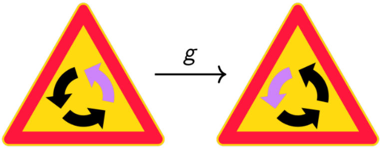
Lie theory - part 3?

Or: Matrix groups

What is a matrix group?



$$\leftrightsquigarrow$$

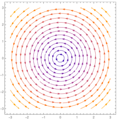
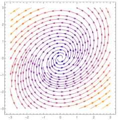
$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- **Matrix group** A subgroup $G \subseteq GL(n)$: closed under multiplication, has an identity, and every element has an inverse
- G is often given by **equations in the matrix entries** (e.g. $\det = 1$)
- **Examples** $GL(n)$, $SL(n)$, $O(n)$, $SO(n)$, $SP(n)$

Why matrices?

Seriously, what is a matrix (via actions)?

Answer 3. A transformation of space, *e.g.*

 $\leftrightsquigarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,  $\leftrightsquigarrow \begin{pmatrix} 1/2 & -1 \\ 1 & 0 \end{pmatrix}$

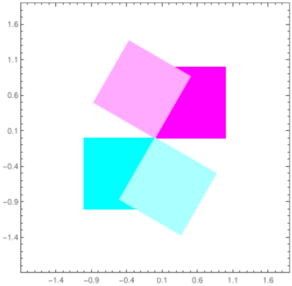
Answer 4. A transformation of shapes, *e.g.*

 $\leftrightsquigarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,  $\leftrightsquigarrow \begin{pmatrix} 1/2 & -1 \\ 1 & 0 \end{pmatrix}$

- **The input** Differential equations and geometry come with families of symmetries (varying continuously)
- **Local view** A smooth symmetry looks linear "to first order" near a point: take its derivative (Jacobian)
- **First conclusion** Derivatives live in $GL(n)$ $\Rightarrow$ matrices are the natural language for infinitesimal symmetry

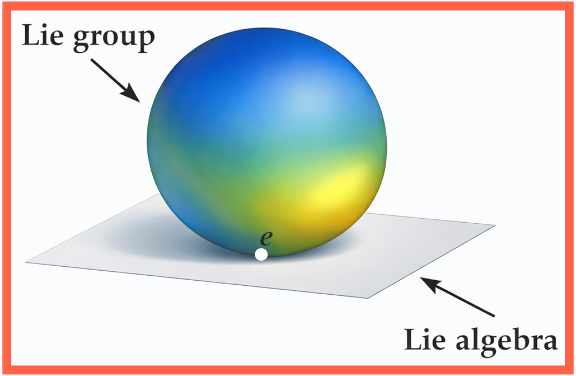
Matrix groups from “preserving structure”

60 rotation matrix $\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$



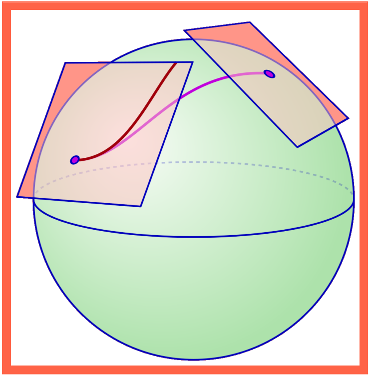
- Same template Pick a structure, then take all linear maps that preserve it
- Examples Length $\Rightarrow O(n)$; volume $\Rightarrow SL(n), SO(n)$; symplectic form $\Rightarrow SP(n)$
- Why this is great Definitions are explicit equations in entries $\Rightarrow$ concrete calculations become possible

The key move: differentiate the defining equations



- Matrix groups are "solutions of equations" inside $GL(n)$
- Differentiate at e (id matrix) The equations linearize $\Rightarrow$ you get a vector space of matrices (the Lie algebra)
- Extra structure Commutators in the group produce a bracket in the algebra $\Rightarrow$ linear algebra $+$ one operation

From matrix groups to Lie groups (modern viewpoint)



- Everyone's playground Matrix groups were the perfect laboratory: explicit, ubiquitous, and computable
- Generalization A Lie group is "a smooth group" without insisting on matrices (a manifold + compatible multiplication)
- Philosophy Even when G is abstract, its infinitesimal behavior is linear (and often represented by matrices anyway)

Thank you for your attention!

I hope that was of some help.