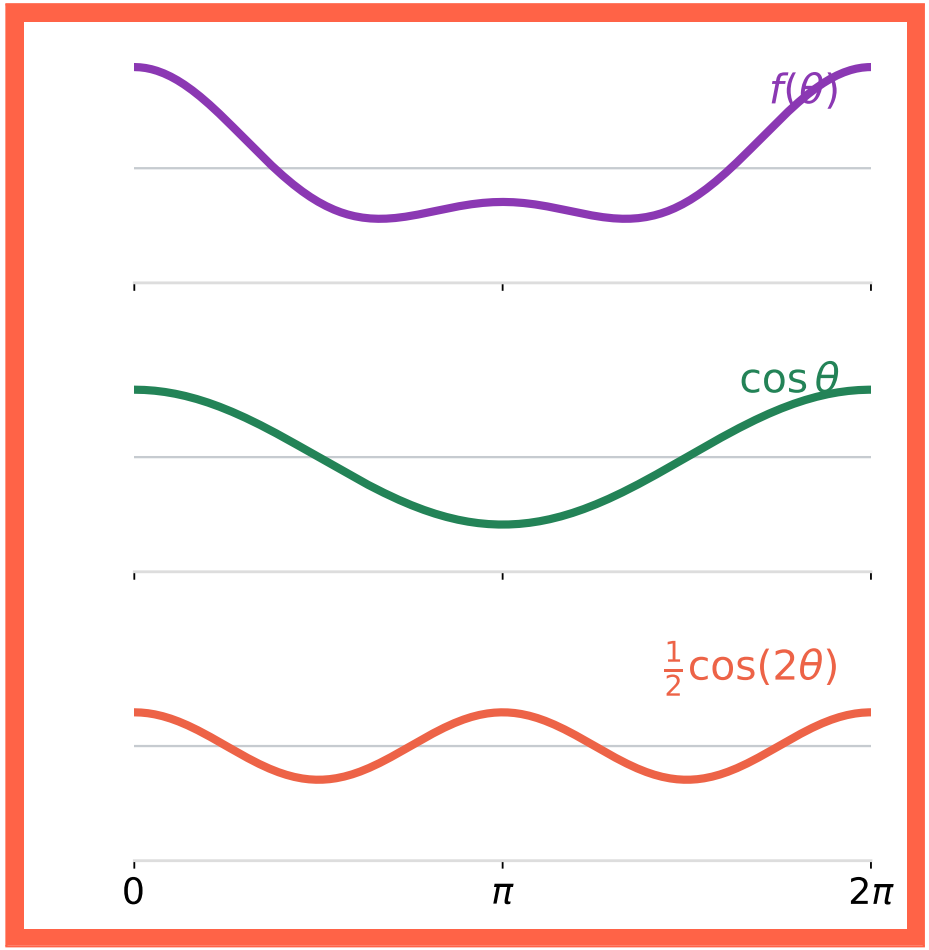
Lie theory - part 30

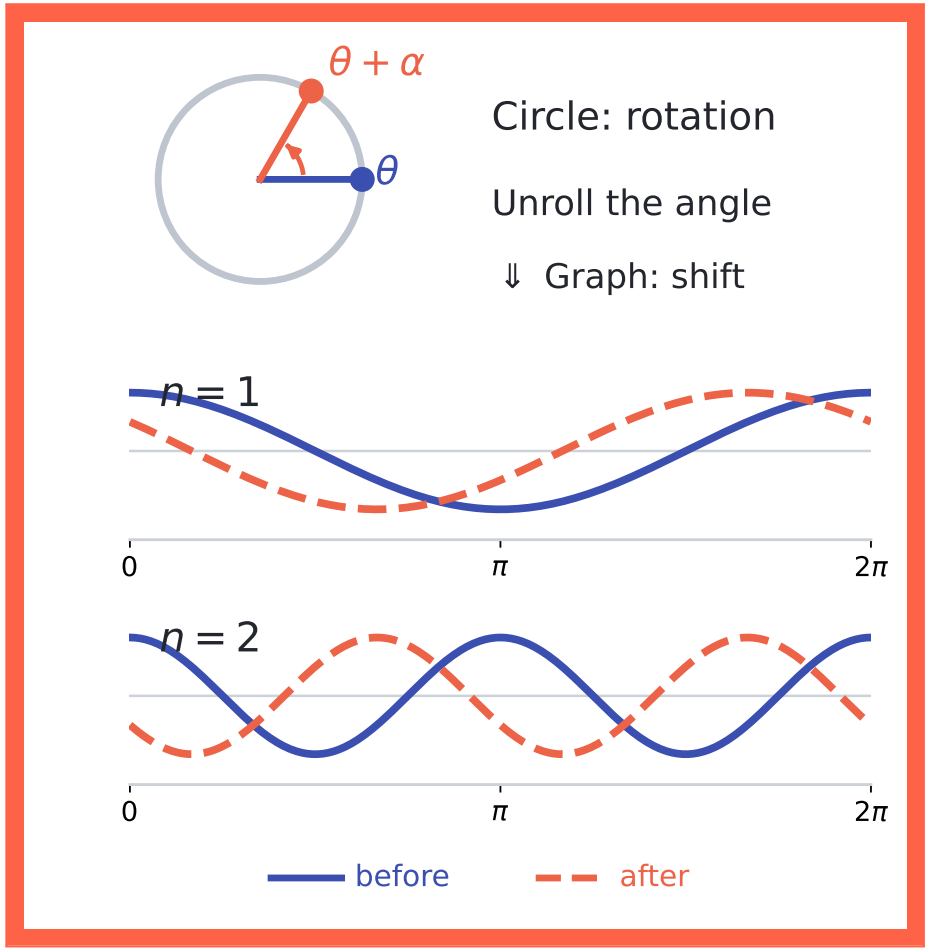
Or: Fourier analysis with symmetry

Fourier's idea: break a signal into waves



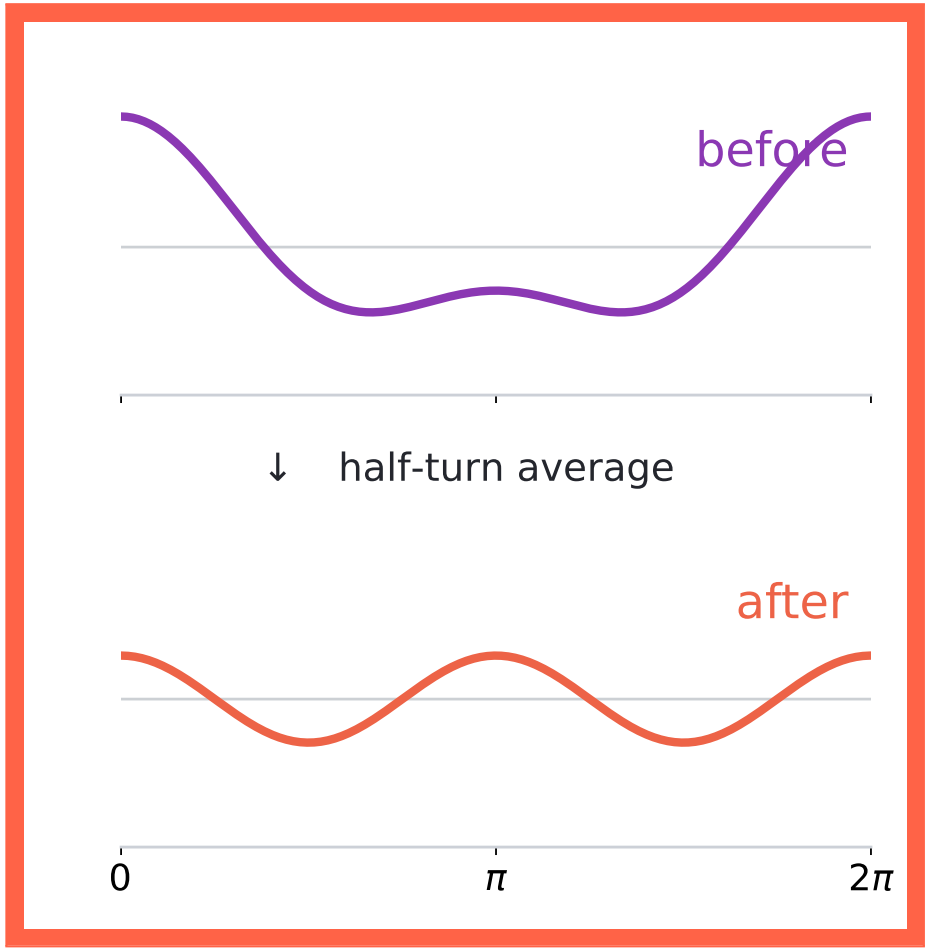
- Example $f(\theta) = \cos\theta + \frac{1}{2}\cos(2\theta)$
- Fourier's idea Break a complicated signal into simple waves
- Why useful? Rotations do not mix different frequencies

Rotation on a circle $U(1) \cong SO(2)$, shift on a graph



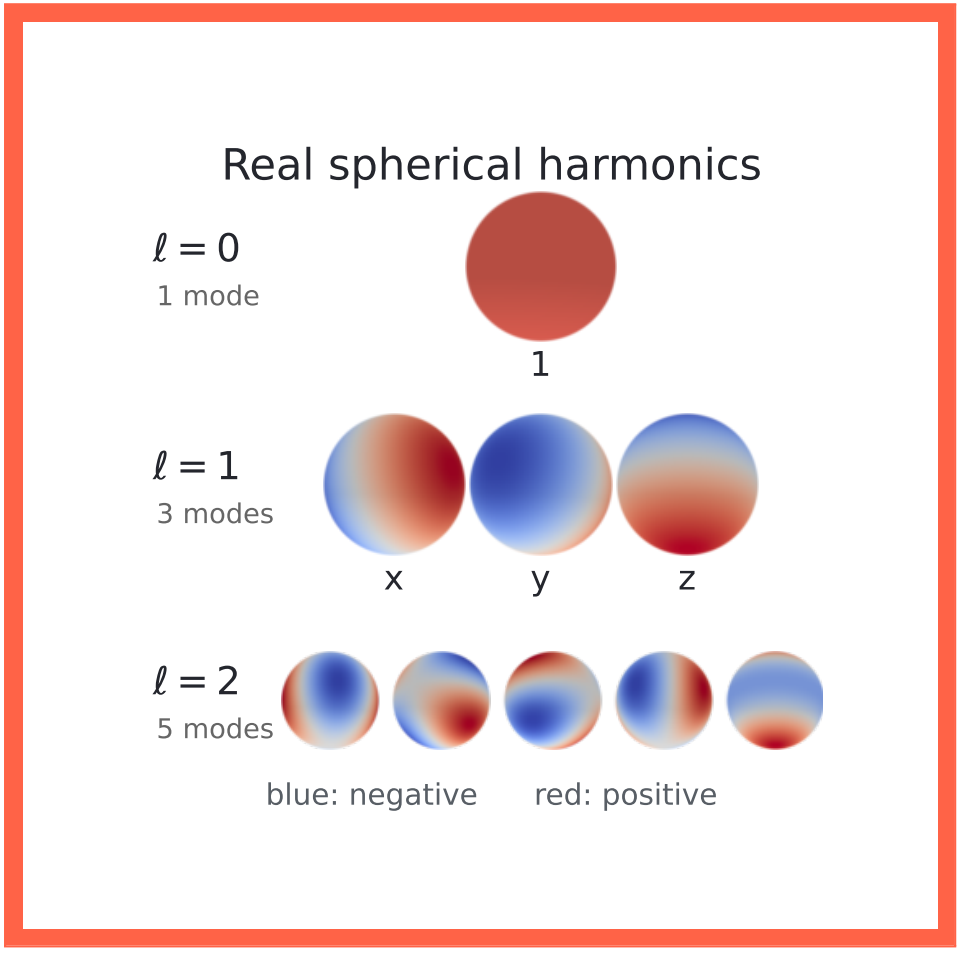
- **Circle** Moving through an angle α is a rotation
- **Graph** Unroll the circle: the rotation looks like a shift
- **Two views** Real sine/cosine mix; complex waves change phase

A symmetry trick: average over rotations



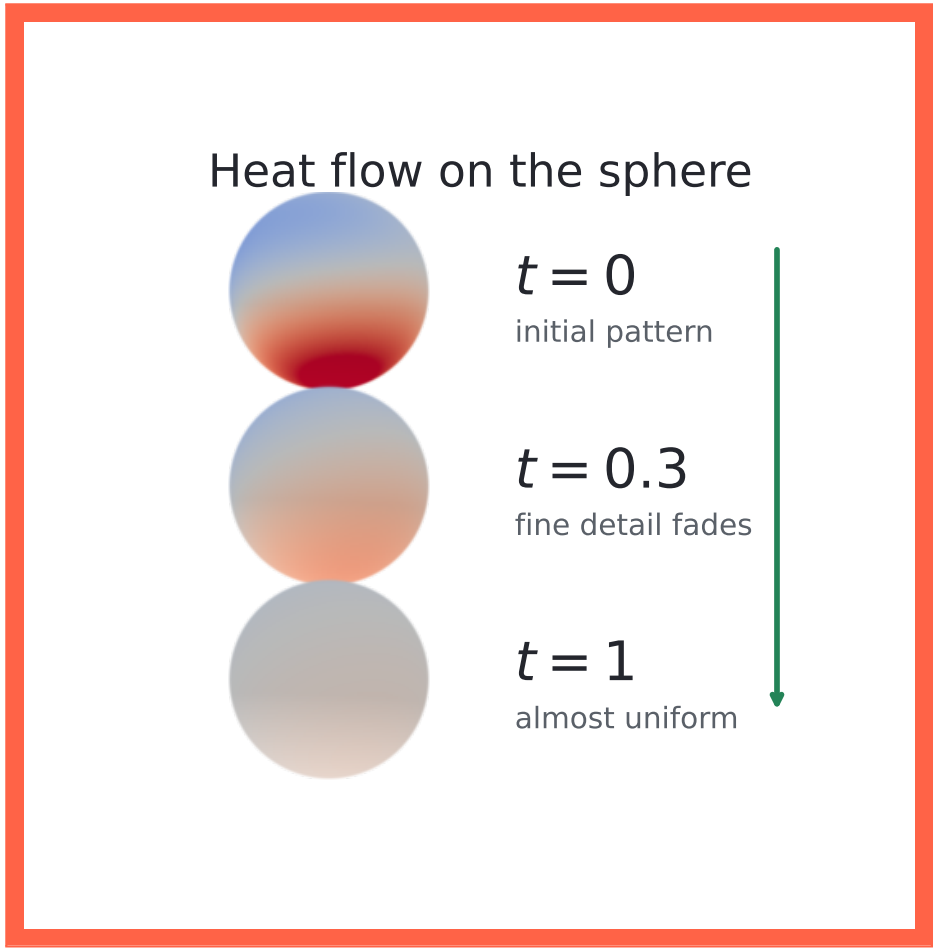
- **Trick** Average the signal with its half-turn shift
- **Formula** $Af(\theta) = \frac{1}{2}\big(f(\theta) + f(\theta+\pi)\big)$
- **Surprise** Odd frequencies vanish; even ones survive

From the circle $SO(2)$ to the sphere $SO(3)$



- Symmetry Rotations of the sphere form $SO(3)$
- Fourier again The basic waves are called spherical harmonics
- New feature Waves come in packets of sizes $1, 3, 5, \ldots$

The payoff: heat spreading on a sphere



- Problem An uneven temperature spreads out over the sphere
- Symmetry Heat flow respects rotations, so packets never mix
- Essentially Solutions form a rep of $SO(3)$, so you can find them using spherical harmonics

Thank you for your attention!

Next time: Angular momentum and quantum spin