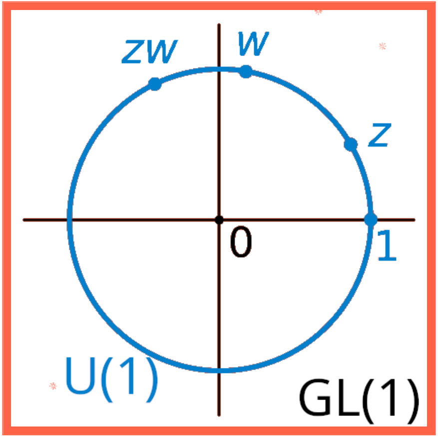
Lie theory - part 4?

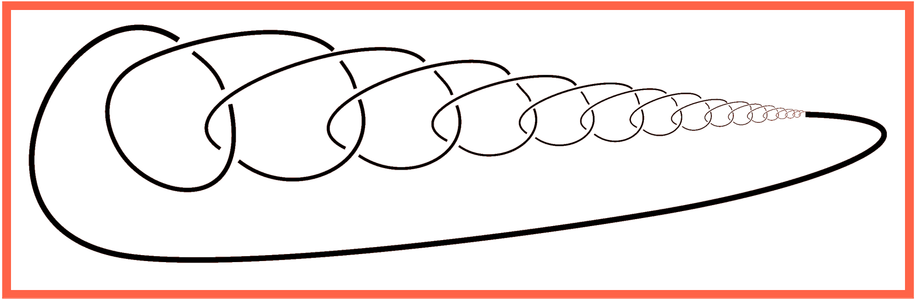
Or: Topology + smoothness

Closed subgroups of $GL(n)$



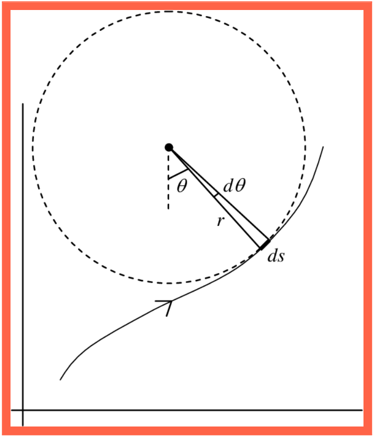
- $GL(n) = GL(n, \mathbb{C})$ is an open subset of $\mathbb{C}^{n^2}$ $\Rightarrow$ it inherits a topology for free
- Matrix (Lie) group (definition) A subgroup $G \subseteq GL(n)$ that is closed
- Why "closed"? Limits of matrices in G stay in G $\Rightarrow$ no "missing boundary points" and better geometry

Smoothness: why it matters



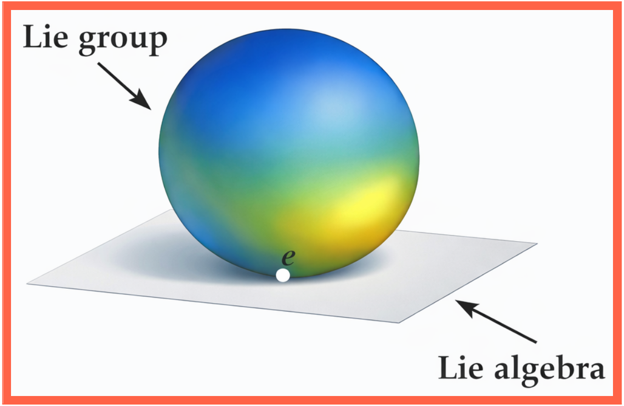
- Topology tells you about continuity (neighbourhoods, convergence, compactness, ...)
- Smoothness lets you differentiate: tangent vectors, Jacobians, differential equations, ...
- Closed-subgroup theorem Matrix Lie groups are automatically smooth
- Non-smooth things are often very nasty...

Curves in a group



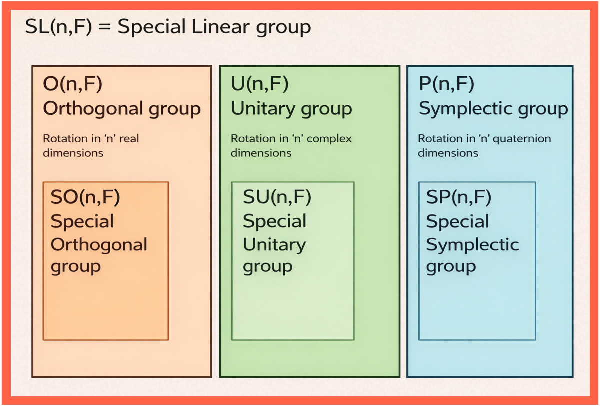
- One idea: Study G by looking at smooth curves $\gamma : (-\varepsilon, \varepsilon) \to G$
- Base point: If $\gamma(0) = e$, then $\gamma(t)$ is a "motion" starting at the identity
- Derivative: The velocity $\gamma'(0)$ is an infinitesimal symmetry $\Rightarrow$ the first linear shadow of G

Tangents at the identity



- Tangent space $T_e G$ is the set of all velocities $\gamma'(0)$ of curves with $\gamma(0) = e$
- For matrix groups $T_e G \subseteq M(n)$: matrices X such that $I + tX + t^2 X \ldots \in G$ for small t; e.g. for $G = GL(n)$ this is $M(n)$ (all matrices)
- Next step This vector space carries a bracket $[X, Y] = XY - YX \Rightarrow$ the Lie algebra of G

Lie groups abstractly



- **Abstract definition** A Lie group is a smooth manifold with smooth multiplication and inversion; almost all I say has 'direct abstract analogs'
- Matrix groups are **prototypical** special cases
- **Example** A torus (e.g. an elliptic curve $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$) is a Lie group, but not a matrix group

Thank you for your attention!

Next time: one-parameter subgroups and exponentials.