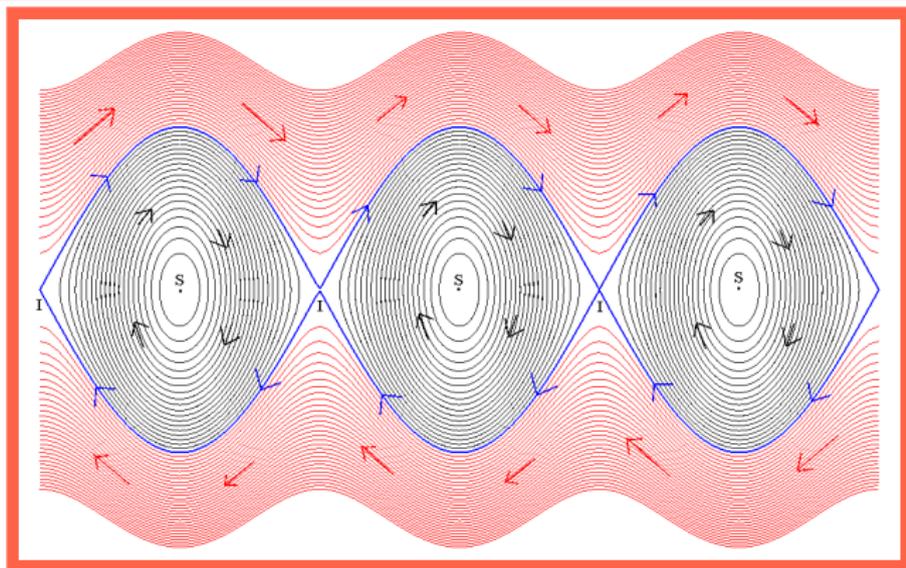


Lie theory - part 5

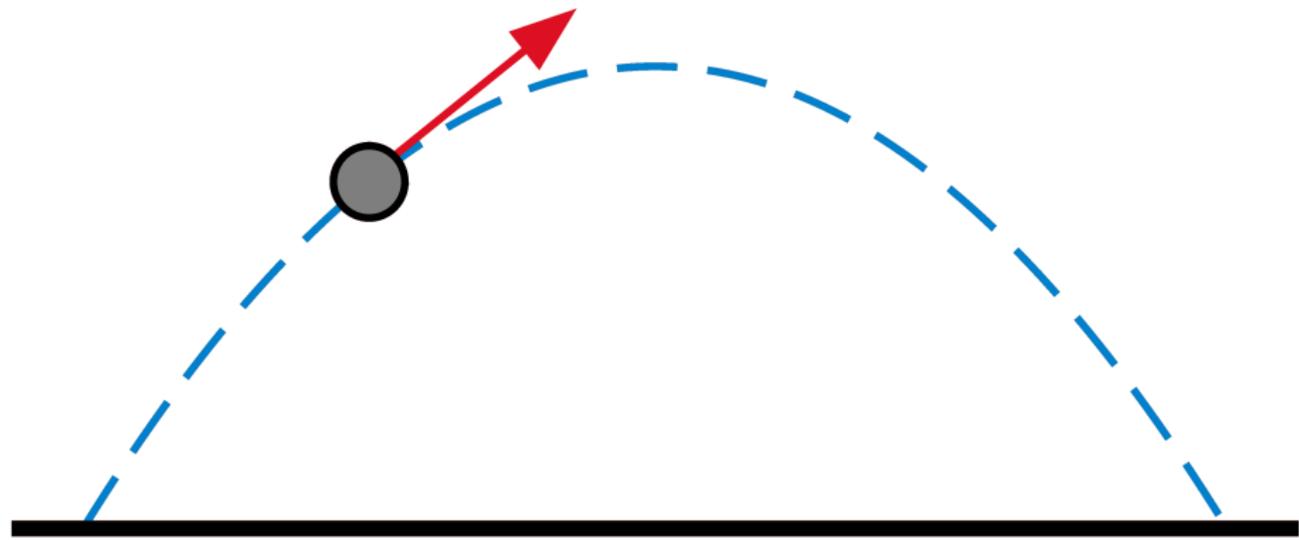
Or: One-parameter subgroups and flows

One-parameter subgroups



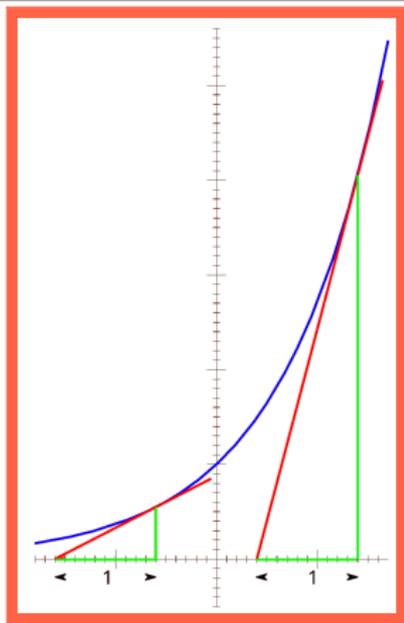
- ▶ **Goal** Curves in G describe continuous symmetries; one-parameter subgroups are the ones compatible with multiplication
- ▶ **Add time = compose** $\phi(s+t) = \phi(s)\phi(t)$ and $\phi(0) = e$: the motion is consistent with the group law
- ▶ **Infinitesimal data** $\phi'(0) \in T_e G = \mathfrak{g}$: the Lie algebra is “velocities at id”

Flows from linear ODEs



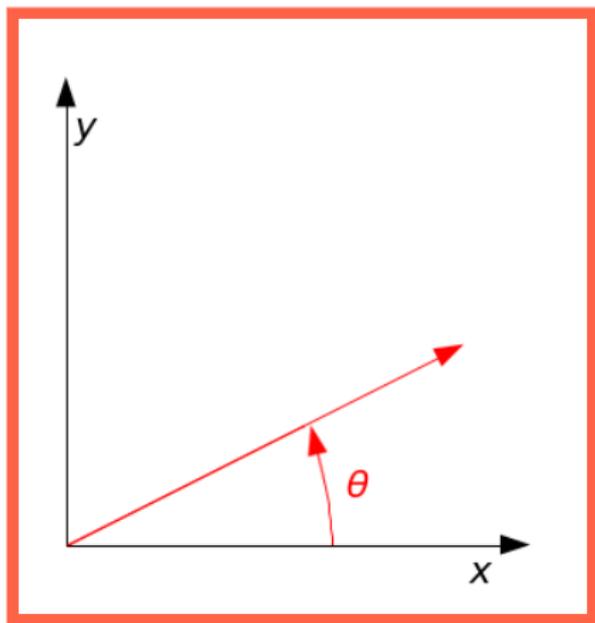
-
- ▶ **Key** $X'(t) = AX(t)$, $A \in M_n(\mathbb{C})$: linear systems generate motion in $GL(n)$
 - ▶ **Add time = compose** $X(0) = I$ and $X(t+s) = X(t)X(s)$
 - ▶ Solving ODE \Rightarrow one-parameter subgroups; A is the infinitesimal generator

The matrix exponential



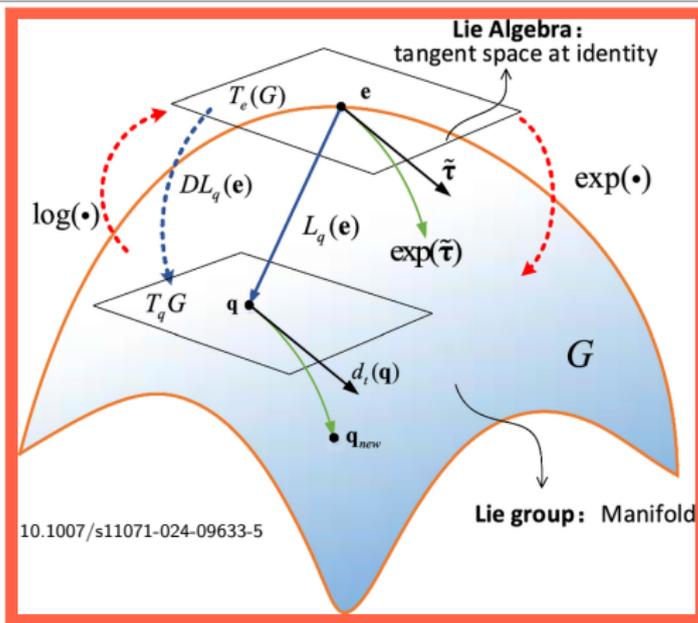
- ▶ From infinitesimal to an actual element $e^A = \sum_{k \geq 0} \frac{A^k}{k!}$ is the universal way to “integrate” a constant generator (respecting the group law)
- ▶ The key map $t \mapsto e^{tA}$ is a one-parameter subgroup in $GL(n)$
- ▶ Check $\frac{d}{dt} e^{tA} = A e^{tA}$ and $e^{0 \cdot A} = I$: derivative at 0 recovers A

Examples; symmetries



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- ▶ Rotations (real slice) In $SO(2)$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ gives $e^{tA} =$ rotation by angle t
 - ▶ Scaling $A = \lambda I$ gives $e^{tA} = e^{\lambda t} I$: uniform expansion / contraction
 - ▶ Shears Nilpotent A (with $A^2 = 0$) gives $e^{tA} = I + tA$: “linear drift”

Why this is the bridge



- ▶ **Question** For $A \in \mathfrak{g} = T_e G$, how do we build an actual curve in G ?
- ▶ **Answer** The exponential does it: $t \mapsto e^{tA}$
- ▶ **Big picture** Locally: $\mathfrak{g} \leftrightarrow$ one-parameter subgroups; next: compute \mathfrak{g} for matrix groups $GL(n)$ and friends

Thank you for your attention!

Next time: the Lie algebra of a matrix Lie group (compute $\mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{su}_n$).