Lie theory - part 6

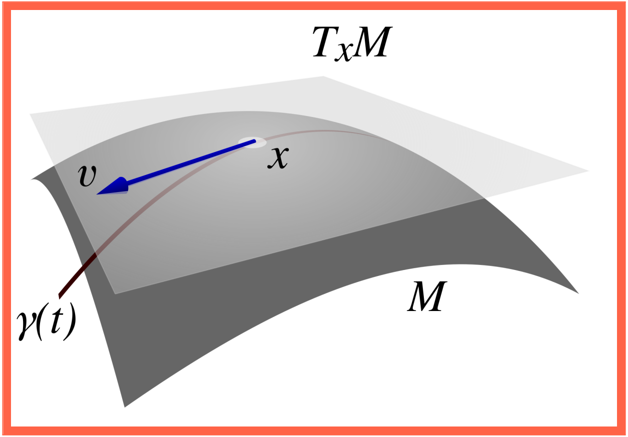
Or: Lie algebras

Why Lie algebras show up

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

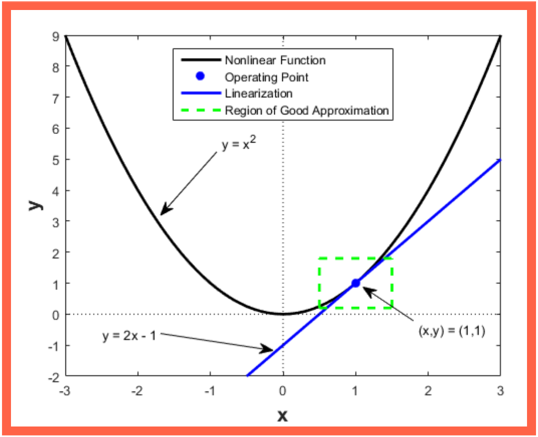
- Big picture G is nonlinear, but its tangent space at the identity is linear
- The point The Lie algebra is the “best linear approximation” of G near e
- Payoff Many group questions become linear algebra questions (equations, dimensions, constraints)

Definition



- Setup Let $G \subseteq GL_n(\mathbb{C})$ be a (complex) matrix Lie group
- Tangent-at-identity $\mathfrak{g} = T_e G \leftrightsquigarrow \{\gamma'(0) \mid \gamma : (-\varepsilon, \varepsilon) \to G,\ \gamma(0) = I\}$
- Intuition $\mathfrak{g}$ = instantaneous directions you can move inside G at I

How to compute $\mathfrak{g}$ in practice



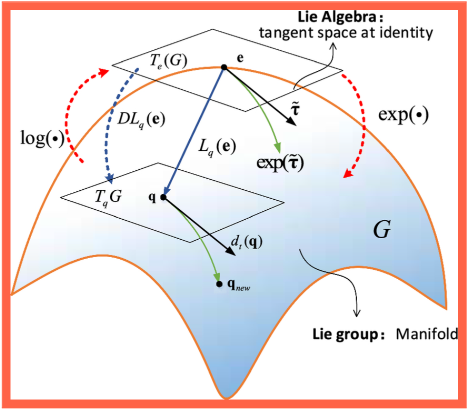
- Rule of thumb Write the defining equations for G, then differentiate at I
- Linearization Nonlinear constraints become linear constraints on $X = \gamma'(0)$
- Result You get a concrete subspace $\mathfrak{g} \subseteq M_n(\mathbb{C})$ "cut out by linear equations"

The classical computations

Lie Group	Lie Algebra
General Linear Group: $GL(n) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$	$\mathfrak{gl}(n) = \{A \in M_n(\mathbb{R})\}$
Special Linear Group: $SL(n) = \{A \in GL_n(\mathbb{R}) : \det A = 1\}$	$\mathfrak{sl}(n) = \{A \in \mathfrak{gl}(n) : \operatorname{tr} A = 0\}$
Special Ortogonal Group: $SO(n) = \{A \in SL(n, \mathbb{R}) : AA^T = I\}$	$\mathfrak{so}(n) = \{A \in \mathfrak{sl}(\mathbb{R}) : A + A^T = 0\}$
Special Unitary Group: $SU(n) = \{A \in SL(n, \mathbb{C}) : A\overline{A}^T = I\}$	$\mathfrak{su}(n) = \{A \in \mathfrak{sl}(n, \mathbb{C}) : A + \overline{A}^T = 0\}$
Symplectic Group: $Sp(n) = \{A \in GL(2n, \mathbb{R}) : AJA^T = J\}$	$\mathfrak{sp}(n) = \{A \in \mathfrak{gl}(2n, \mathbb{R}) : AJ + JA^T = 0\}$
Group of isometries of (M, g): $G = \{\phi : M \to M \mid \phi \text{ preserves } g\}$	$\mathfrak{g} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X g = 0\}$
Group of symplectomorphisms of (M, ω): $G = \{\phi : M \to M \mid \phi^* \omega = \omega\}$	$\mathfrak{g} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X \omega = 0\}$

- $GL_n(\mathbb{C})$ det $\neq 0$ is an open condition, so there is no tangent constraint at I; hence $\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$
- Key linearization $\det(I + tX) = 1 + t\, tr(X) + O(t^2)$
- $SL_n(\mathbb{C})$ If $\det(g(t)) \equiv 1$, then the derivative is 0, so $tr(g'(0)) = 0$; hence $\mathfrak{sl}_n(\mathbb{C}) = \{X \mid tr(X) = 0\}$
- Other matrix groups (same game) Linearize the defining equation at I: $SO_n : \ X^T + X = 0,\ Sp_{2n} : \ X^T J + JX = 0,\ U_n : \ X^* + X = 0$ etc.

Back to exponentials



- Direction → motion If $X \in \mathfrak{g}$, then $t \mapsto e^{tX}$ stays inside G
- Interpretation The exponential turns an "allowed infinitesimal move" into a genuine one-parameter subgroup
- Next step Now we can ask: what extra structure lives on $\mathfrak{g}$? (that's where the bracket enters)

Thank you for your attention!

Next time: where the Lie bracket comes from (commutators and conjugation)