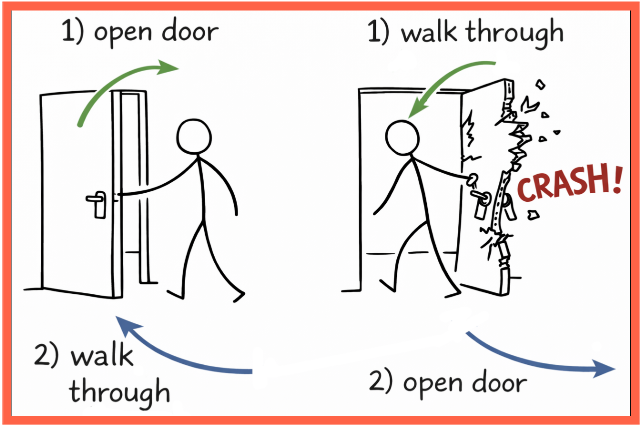
Lie theory - part 7

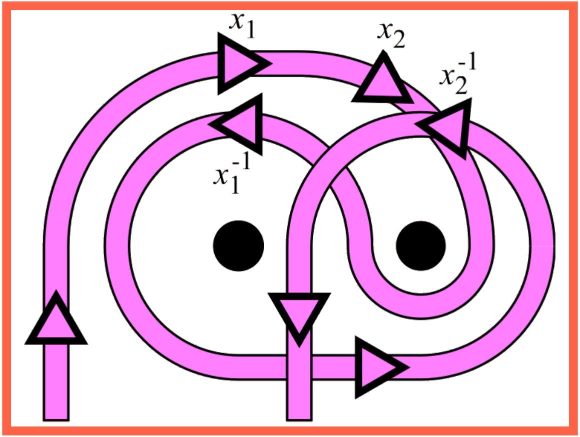
Or: Lie brackets

Why we need a bracket at all



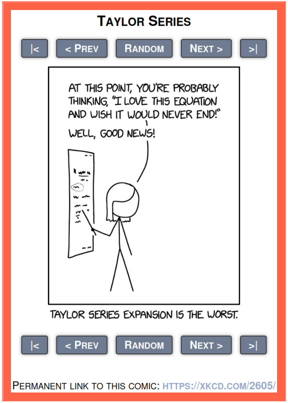
- Big picture $\mathfrak{g} = T_e G$ is linear, but G is usually not commutative
- Missing information The bracket $[\,,\,]$ is the extra structure on $\mathfrak{g}$ that remembers “how noncommutative G is”
- Promise Once you have $[\,,\,]$, $\mathfrak{g}$ becomes a calculator for local group behavior

The commutator: a noncommutativity meter



- **Definition** $[g, h] = ghg^{-1}h^{-1}$ (equals e iff $gh = hg$)
- **Interpretation** It is the "error term" you pick up when swapping g and h
- **Lie idea** Look at $[g, h]$ for g, h *near* e, and read off the first nontrivial term

Infinitesimal commutator $\Rightarrow$ the bracket

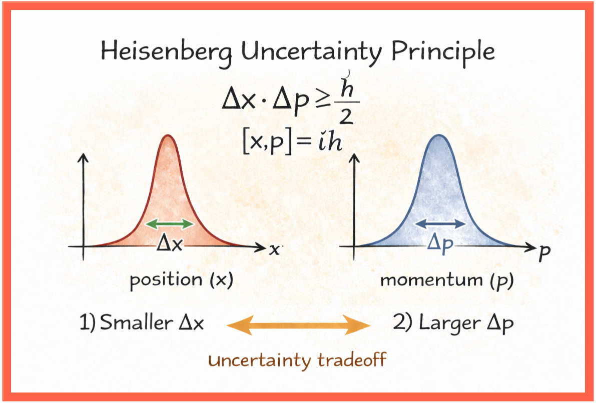


- **Plug in exponentials** Take $g(t) = e^{tX}$ and $h(t) = e^{tY}$ with $X, Y \in \mathfrak{g}$
- **First surprise** The commutator is *invisible to first order*: $[g(t), h(t)] = I + 0 \cdot t + x \cdot t^2 + O(t^3)$
- **Definition by the t^2-term** The coefficient of t^2 is $[X, Y]$ (for matrices: $[X, Y] = XY - YX$)

Differentiating conjugation: the adjoint story

- Conjugation acts $c_g(h) = ghg^{-1}$ moves elements around inside G
- Adjoint action Differentiating at e gives a linear map $Ad_g : \mathfrak{g} \to \mathfrak{g}$
- Bracket as derivative Differentiating $Ad_{e^{tX}}$ at $t = 0$ gives $ad_X(Y) = [X, Y]$

What the bracket does for you



- Core formula For matrix groups, $[X, Y] = XY - YX$ and $\mathfrak{g}$ is closed under it
- Structural rules Bilinear, antisymmetric, and satisfies Jacobi: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$
- Bigger picture Subgroups, symmetries, and "what is preserved" become questions about subalgebras and homomorphisms

Thank you for your attention!

Next time: Subgroups/symmetries $\Rightarrow$ linear constraints