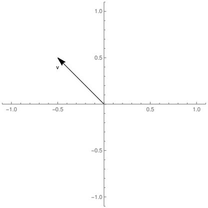
What is...a vector space?

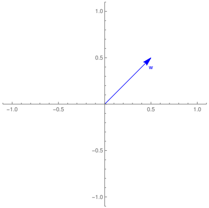
Or: Geometry à la Descartes.

First example – arrows

A collection of arrows
starting at the origin:
Yes, this is a vector space – $\mathbb{R}^2$

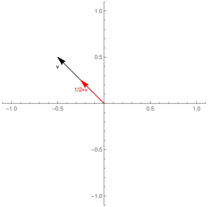


or

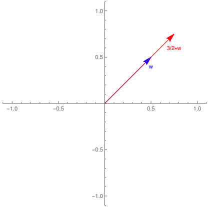


or...

We can scale arrows:

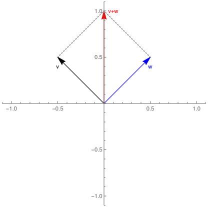


or

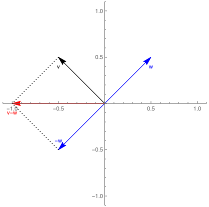


or...

We can add
(and subtract) arrows:



or



or...

Second example – matrices

A collection of matrices
over $\mathbb{R}$ and 2x2
Yes, this is a vector space – 2x2 matrices

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{or} \quad N = \begin{pmatrix} -1 & 1 \\ 2 & 1/2 \end{pmatrix} \quad \text{or...}$$

We can scale matrices:

$$5 \cdot M = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix} \quad \text{or} \quad -1 \cdot N = \begin{pmatrix} 1 & -1 \\ -2 & -1/2 \end{pmatrix} \quad \text{or...}$$

We can add
(and subtract) matrices:

$$M + N = \begin{pmatrix} 0 & 3 \\ 5 & 9/2 \end{pmatrix} \quad \text{or} \quad M - N = \begin{pmatrix} 2 & 1 \\ 1 & 7/2 \end{pmatrix} \quad \text{or...}$$

Third example – polynomials

A collection of polynomials
over $\mathbb{R}$ and in X
Yes, this is a vector space – $\mathbb{R}[X]$

$$P = X^2 + 5X \quad \text{or} \quad Q = -10X^3 + 2 \quad \text{or...}$$

We can scale polynomials: $5 \cdot P = 5X^2 + 25X$ or $-1 \cdot Q = 10X^3 - 2$ or...

We can add (and subtract) polynomials: $P + Q = -10X^3 + X^2 + 5X + 2$ or $P - Q = 10X^3 + X^2 + 5X - 2$ or...

For completeness: A formal definition.

A vector space V (over some field $\mathbb{K}$) is a set, whose elements are called vectors, together with two operations:

- Scalar multiplication $\lambda \cdot v$ of vectors $v \in V$ by a scalar $\lambda \in \mathbb{K}$
- Addition (and subtraction) $v + w$ of vectors $v, w \in V$

These operations should satisfy:

Associativity	$(v + w) + x = v + (w + x)$
Commutativity	$v + w = w + v$
Identity 1	$\exists 0$ such that $v + 0 = v = 0 + v$
Inverses	$\exists -v$ such that $v + (-v) = 0 = (-v) + v$
Compatibility	$(\lambda\mu) \cdot v = \lambda \cdot (\mu \cdot v)$
Identity 2	$1 \cdot v = v$
Distributivity 1	$\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$
Distributivity 2	$(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$

Vector spaces can be tiny or huge

2x2 matrices over $\mathbb{Z}/2\mathbb{Z}$	$\left\{\begin{matrix} \begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix} \\ \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix} \\ \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\1&1\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix} \\ \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix}, \begin{pmatrix}1&1\\1&1\end{pmatrix} \end{matrix}\right\}$	Only 16 elements
Functions $f : \mathbb{R} \to \mathbb{R}$	$(f+g)(x) = f(x) + g(x)$ $(\lambda \cdot f)(x) = \lambda f(x)$	Infinite dimensional

Thank you for your attention!

I hope that was of some help.