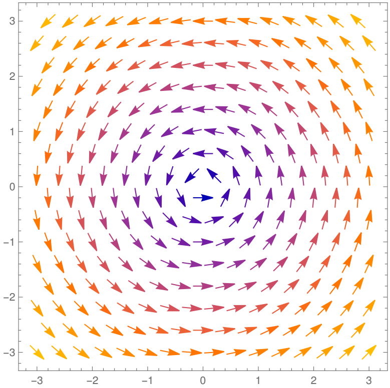
What are...eigenvalues and eigenvectors?

Or: Linear fixed points.

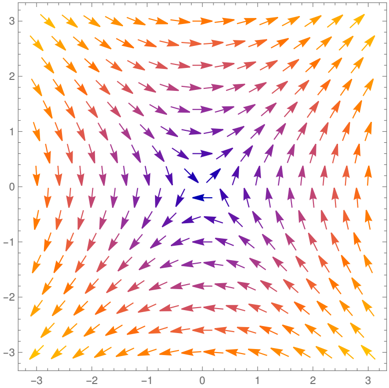
Fixed vectors.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



no fixed vector

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



fixed vectors

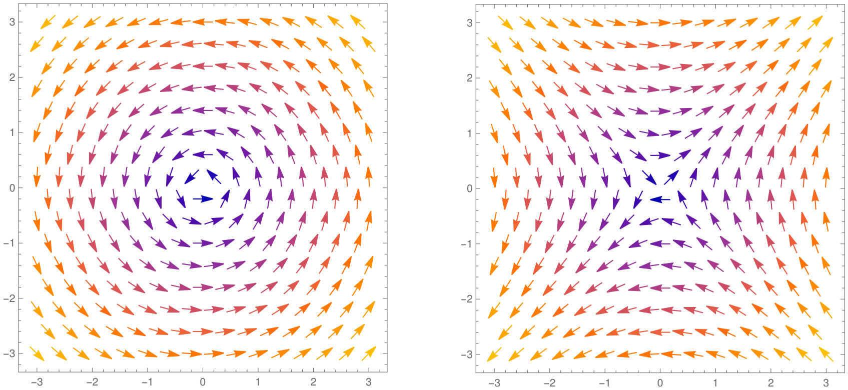
What does it mean to be fixed by a matrix?

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

No solution in $\mathbb{R}^2$ since $-y = \lambda x$ and $x = \lambda y$ implies $\lambda^2 x = -x$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

Solutions in $\mathbb{R}^2$: ($\lambda = 1$ and $x = y$) or ($\lambda = -1$ and $x = -y$).

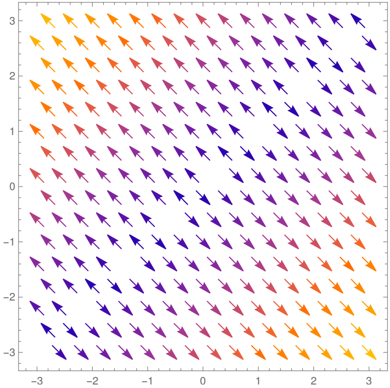


Solutions to "matrix times vector = scalar vector" are called eigenvectors; the scalar is called an eigenvalue.

More example!

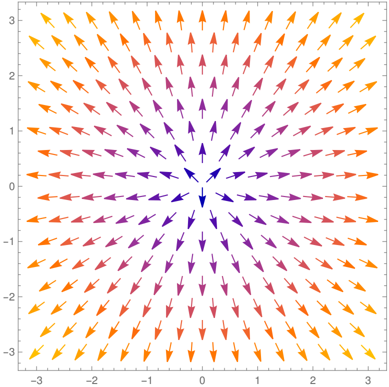
$$\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x-y\\ y-x\end{pmatrix} = \lambda\begin{pmatrix}x\\ y\end{pmatrix}$$

Solutions in $\mathbb{R}^2$: ($\lambda = 2$ and $x = -y$) or ($\lambda = 0$ and $x = y$).



$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix} = \lambda\begin{pmatrix}x\\ y\end{pmatrix}$$

Solutions in $\mathbb{R}^2$: $\lambda = 1$ and "every vector".



The zero vector is a boring solution, so it is always excluded.
The zero eigenvalue is interesting.

For completeness: A formal definition.

An eigenvalue λ of a nxn matrix M is a scalar in the underlying ground field such that there exists a vector $v \neq 0$ with $Mv = \lambda \cdot v$.

Any such v is called an eigenvector of M of eigenvalue λ.

Another thing that is cool about eigenvalues and eigenvectors.

Try to find M^{100} for some matrix M. For example:

$$M = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}, M^2 = \begin{pmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{pmatrix}, ..., M^{100} \approx \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}.$$

Eigenvalues are 1 and 0.5, eigenvectors are $\approx (0.832, 0.555)$ and $(-0.707, 0.707)$.

Now calculate

$$M^{100} \approx \begin{pmatrix} 0.832 & -0.707 \\ 0.555 & 0.707 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.832 & -0.707 \\ 0.555 & 0.707 \end{pmatrix}^{-1} \approx \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}.$$

Why? We see that in another video.

Thank you for your attention!

I hope that was of some help.