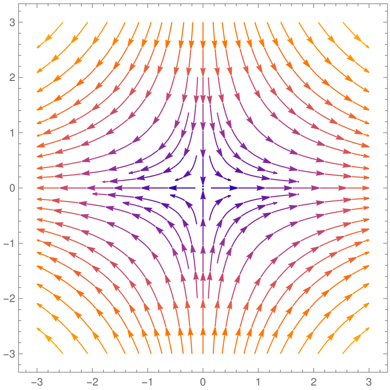
What is...diagonalization?

Or: Finding the right coordinate system.

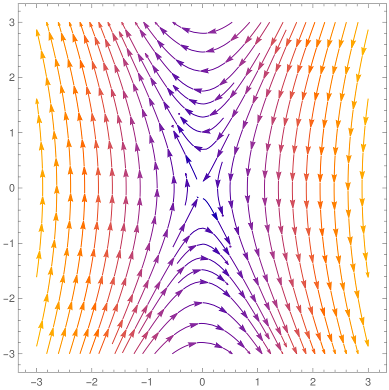
Making axes eigenvectors

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



The *xy*-axes are eigenvectors

$$\begin{pmatrix} 0 & -1/2 \\ -2 & 0 \end{pmatrix}$$



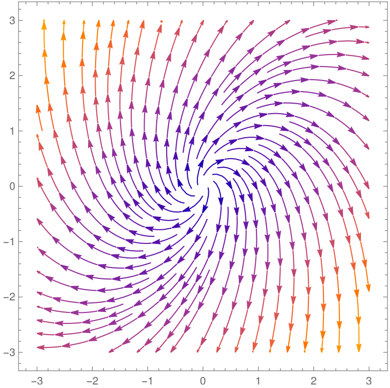
The *xy*-axes are not eigenvectors

Rotate, reflect and scale

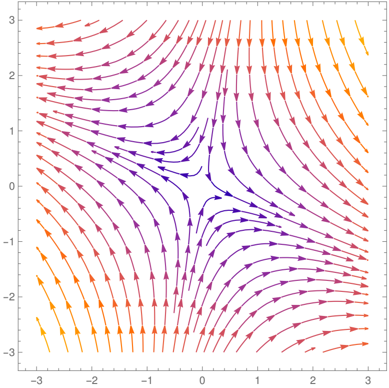
$\begin{pmatrix} 0 & -1/2 \\ -2 & 0 \end{pmatrix}$ has eigenvectors $(1/2, -1)$, $(1/2, 1)$. Take $P = \begin{pmatrix} 1/2 & -1 \\ 1/2 & 1 \end{pmatrix}$.

$$P(a) = \begin{pmatrix} a/2 + (1-a) & -a \\ a/2 & 1 \end{pmatrix}$$

$$P(2/3)^{-1} \begin{pmatrix} 0 & -1/2 \\ -2 & 0 \end{pmatrix} P(2/3)$$



$P(1)$ moves everything in place



$P(2/3)$ gets us almost there

How can we check whether some matrix is diagonalizable?

To check whether an n-n matrix M is diagonalizable we:

- First calculate the characteristic polynomial $p(X)$.
- Check whether $p(X)$ has n distinct roots (eigenvalues λ) – if yes, we are in business.
- If no, then we need to find the eigenvectors by solving

$$(\lambda - M)v = 0.$$

- If we get n linear independent solutions, then M is diagonalizable, and otherwise it is not.

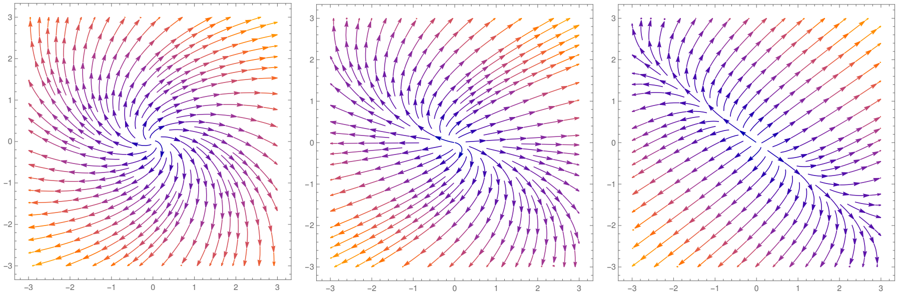
For completeness: A formal definition.

A matrix M (over some ground field) is called diagonalizable if there exists an invertible matrix P such that $P^{-1}MP$ is diagonal.

- This happens if and only if there exists a basis given by eigenvectors of M.

Are all matrices diagonalizable? Well, almost all...

$$\begin{pmatrix} 1 & 1 \\ a & 1 \end{pmatrix} \begin{cases} \text{not dia over } \mathbb{R}\text{, but over } \mathbb{C} \text{ if } a < 0, \\ \text{not dia over } \mathbb{R} \text{ or } \mathbb{C} \text{ if } a = 0, \\ \text{dia over } \mathbb{R} \text{ and } \mathbb{C} \text{ if } a > 0. \end{cases}$$



$a = -0.5$ $a = 0$ $a = 0.5$

Almost all matrices are diagonalizable – over $\mathbb{C}$ – we will see this when we generalize this notion to the Jordan normal form.

Thank you for your attention!

I hope that was of some help.