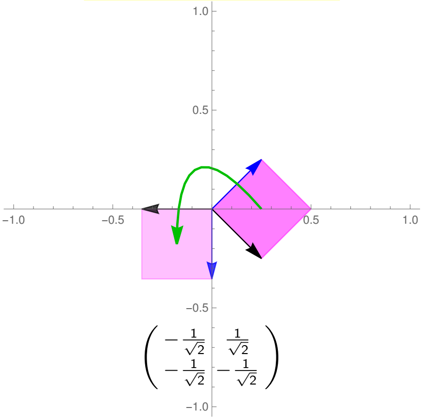
What is...an isometry?

Or: Tessellations of my walls

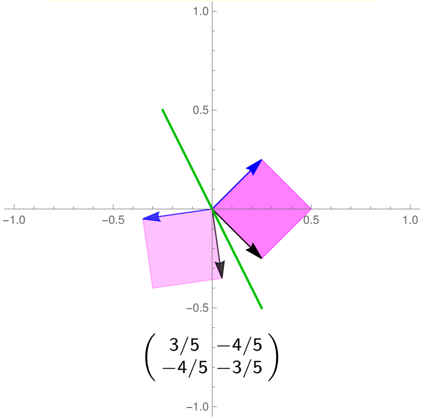
Isometry = a length preserving action

225° rotation preserves the length



$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Reflecting along a line preserves the length



$$\begin{pmatrix} 3/5 & -4/5 \\ -4/5 & -3/5 \end{pmatrix}$$

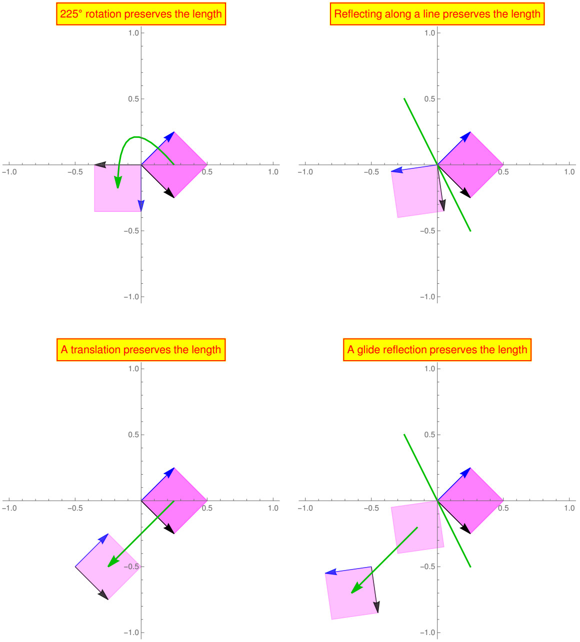
Matrices that preserve length

$$225^\circ \text{ rotation: } \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \text{transpose=inverse: } \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\text{Reflection along a line: } \begin{pmatrix} 3/5 & -4/5 \\ -4/5 & -3/5 \end{pmatrix}, \quad \text{transpose=inverse: } \begin{pmatrix} 3/5 & -4/5 \\ -4/5 & -3/5 \end{pmatrix}$$

Transpose=inverse holds for all these length preserving maps

All Euclidean plane isometries



For completeness: A formal definition

Given two inner product spaces V and W, a (linear) isometry is a (linear) map $M\colon V \to W$ that preserves the inner product $\langle v, v \rangle = \langle Mv, Mv \rangle$

Important facts:

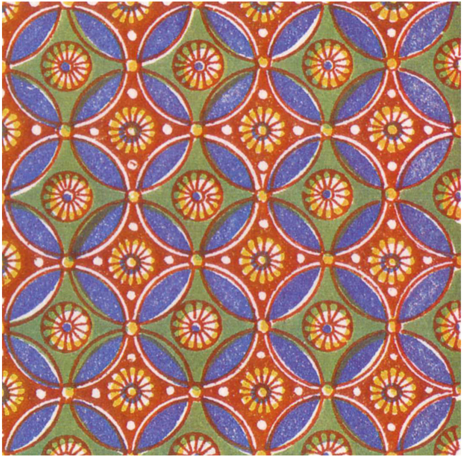
(a) $\langle v, w \rangle = \langle Mv, Mw \rangle$ also holds, so isometries preserve angles as well

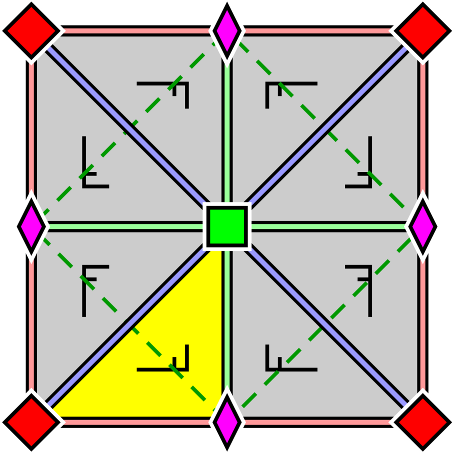
(b) For $V = W$ the condition $\langle v, v \rangle = \langle Mv, Mv \rangle$ is equivalent to $MM^* = M^*M = \mathrm{id}$

(c) For $V = W = \mathbb{R}^n$ with Euclidean inner product we have: M is a linear isometry if and only if $M^T = M^{-1}$ Orthogonal

(d) For $V = W = \mathbb{C}^n$ with Euclidean inner product we have: M is a linear isometry if and only if $\overline{M^T} = M^{-1}$ Unitary

Wallpaper groups (there are 17 - here p4m)





Can you spot all symmetries?

Thank you for your attention!

I hope that was of some help.