What is...Gaussian elimination?

Or: How to find intersections.

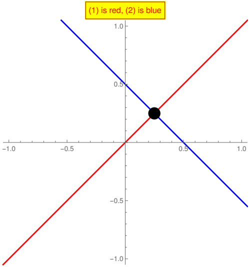
Swapping two rows

$$\begin{cases} 1x - 1y = 0 & (1) \\ 1x + 1y = 1/2 & (2) \end{cases}$$

or equivalently

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 1/2 \end{array}\right)$$

$\leftrightsquigarrow$

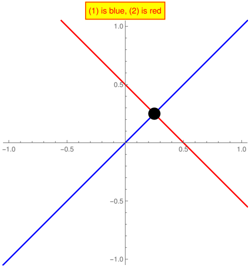


$$\begin{cases} 1x + 1y = 1/2 & (1) \\ 1x - 1y = 0 & (2) \end{cases}$$

or equivalently

$$\left(\begin{array}{cc|c} 1 & 1 & 1/2 \\ 1 & -1 & 0 \end{array}\right)$$

$\leftrightsquigarrow$



The intersection does not change

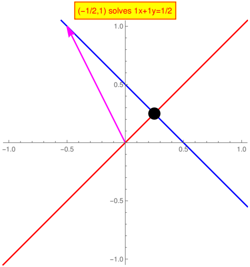
Non-zero scalar times a row

$$\begin{cases} 1x - 1y = 0 & (1) \\ 1x + 1y = 1/2 & (2) \end{cases}$$

or equivalently

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 1/2 \end{array} \right)$$

$\leftrightsquigarrow$

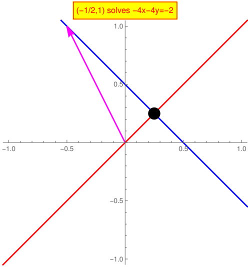


$$\begin{cases} 1x - 1y = 0 & (1) \\ -4x - 4y = -2 & (2) \end{cases}$$

or equivalently

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ -4 & -4 & -2 \end{array} \right)$$

$\leftrightsquigarrow$



The intersection does not change

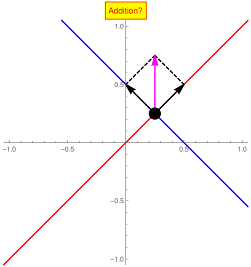
Row plus a row

$$\begin{cases} 1x - 1y = 0 & (1) \\ 1x + 1y = 1/2 & (2) \end{cases}$$

or equivalently

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 1/2 \end{array}\right)$$

$\leftrightsquigarrow$

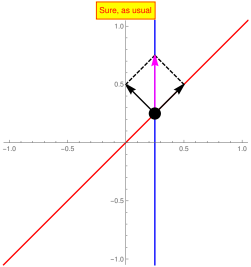


$$\begin{cases} 1x - 1y = 0 & (1) \\ 2x + 0y = 1/2 & (2) \end{cases}$$

or equivalently

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 0 & 1/2 \end{array}\right)$$

$\leftrightsquigarrow$



The intersection does not change

For completeness: A formal definition.

Gaussian elimination is a set of operations on a matrix:

(a) Swapping two rows

(b) Multiplying a row by a nonzero scalar

(c) Adding one row to another row

Important facts:

(a) Gaussian elimination transforms any matrix into staircase form, *e.g.*:

$$\begin{pmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) Row-equivalent matrices have the same rank

(c) Gaussian elimination transforms any system of linear equations into a one which is easy to solve and has the same solutions, *e.g.*:

$$\begin{cases} 2x + y - z = 8 & (1) \\ -3x - y + 2z = -11 & (2) \\ -2x + y + 2z = -3 & (3) \end{cases} \rightsquigarrow \begin{cases} 1x + 0y + 0z = 2 & (1) \\ 0x + 1y + 0z = 3 & (2) \\ 0x + 0y + 1z = -1 & (3) \end{cases}$$

Calculating the inverse matrix

$$M = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Perform Gaussian elimination:

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right)$$

We get

$$M^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

Thank you for your attention!

I hope that was of some help.