What is...the LU decomposition?

Or: Lower-upper.

Having triangles is useful

Lower L and upper U triangular matrices are easy, *e.g.*:

- It is easy to find eigenvalues, compute determinants *etc.*

$$U = \begin{pmatrix} 1 & A & B \\ 0 & 2 & C \\ 0 & 0 & 3 \end{pmatrix} \rightsquigarrow \det(U) = 1 \cdot 2 \cdot 3 = 6$$

- Solving linear equations is straightforward:

$$\begin{pmatrix} 1 & A & B \\ 0 & 2 & C \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightsquigarrow \begin{cases} z = 1/3\, c \\ y = 1/2\,(b - Cz) \\ x = 1/1\,(a - Ay - Bz) \end{cases}$$

- Inversion is clear:

$$U^{-1} = \begin{pmatrix} 1/1 & -A/(1 \cdot 2) & (AC - B2)/(1 \cdot 2 \cdot 3) \\ 0 & 1/2 & -C/(2 \cdot 3) \\ 0 & 0 & 1/3 \end{pmatrix}$$

Question. Can we always factor a matrix as $M = LU$? Almost, as we will see.

Let look at examples, preferring L to be normalized

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{pmatrix} = LU$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} ?$$

Well, instead we solve

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \& \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Slight catch

You might need a permutation matrix to make things work, *e.g.*

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} b & c \\ 0 & d \end{pmatrix} = \begin{pmatrix} b & c \\ ab & ac + d \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Instead:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

In other words, we should expect to have

$$M = PLU$$

For completeness: A formal definition.

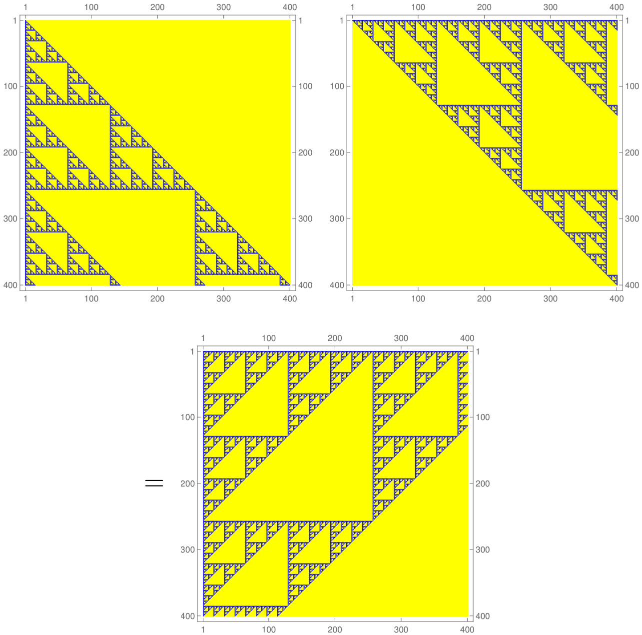
A PLU decomposition of a square matrix M is $M = PLU$, where...

- P is a permutation matrix
- L is lower triangular with 1s on the diagonal
- U is upper triangular

Important facts:

- PLU decompositions always exist and are unique
- Solving $Mx = b$ is equivalent to solving $Ly = Pb$ and $Ux = y$
- The determinant of M is $\det(M) = \pm r_{11}...r_{nn}$, with the sign depending on P

Pascals triangle modulo 2 in its PLU decomposition



Thank you for your attention!

I hope that was of some help.