What is...the symmetric algebra?

Or: Polynomials in vector spaces.

Let us look at polynomials

$\cdot$	bX_1	$+$ dX_2
aX_1	abX_1X_1	adX_1X_2
$+$ cX_2	bcX_2X_1	cdX_2X_2

$\mathrm{perm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad + bc$ $\qquad$ $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$abX_1^2 + cdX_2^2$ $\qquad$ 0

$(ad + bc)X_1X_2$ $\qquad$ $(ad - bc)X_1X_2$

Many similarities, but two crucial differences:

- Diagonals survive *vs.* diagonals are annihilated
- Commuting variables *vs.* anticommuting variables

The permanent

How to get

$$\mathrm{perm}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = afh + aei + bfg + bdi + ceg + cdh?$$

Multiply polynomials:

$$(aX_1 + dX_2 + gX_3)(bX_1 + eX_2 + hX_3)(cX_1 + fX_2 + iX_3) =$$
$$(afh + aei + bfg + bdi + ceg + cdh)X_1X_2X_3$$
$$+\text{a lot of other summands}$$

This was a calculation in degree 3 of the symmetric algebra $\mathrm{Sym}(X_1, X_2, X_3) = \mathbb{R}\langle X_1, X_2, X_3\rangle/(X_iX_j = X_jX_i)$.

Lets count dimensions

Write $\mathrm{Sym}^k(X_1, X_2, X_3)$ for polynomials of degree k in $\mathrm{Sym}(X_1, X_2, X_3)$.

- $\mathrm{Sym}^0(X_1, X_2, X_3)$ is spanned by $\{1\}$
  $\dim \mathrm{Sym}^0(X_1, X_2, X_3) = \binom{3}{0} = 1$
- $\mathrm{Sym}^1(X_1, X_2, X_3)$ is spanned by $\{X_1, X_2, X_3\}$
  $\dim \mathrm{Sym}^1(X_1, X_2, X_3) = \binom{3+1-1}{1} = 3$
- $\mathrm{Sym}^2(X_1, X_2, X_3)$ is spanned by $\{X_1^2, X_2^2, X_3^2, X_1X_2, X_1X_3, X_2X_3\}$
  $\dim \mathrm{Sym}^2(X_1, X_2, X_3) = \binom{3+2-1}{2} = 6$
- $\mathrm{Sym}^3(X_1, X_2, X_3)$ is spanned by $\{X_1^3, X_2^3, X_3^3, X_1^2X_2, X_1^2X_3, X_1X_2^2, X_2^2X_3, X_1X_3^2, X_2X_3^2, X_1X_2X_3\}$
  $\dim \mathrm{Sym}^3(X_1, X_2, X_3) = \binom{3+3-1}{3} = 10$
- All others are $\binom{3+k-1}{k}$, and the total dimension is ∞
  $\dim \mathrm{Sym}(X_1, X_2, X_3) = \infty$

$\dim \mathrm{Ext}^k(X_1, ..., X_n) = \binom{n}{k}$
$\dim \mathrm{Ext}(X_1, ..., X_n) = 2^n$

$\dim \mathrm{Sym}^k(X_1, ..., X_n) = \binom{n+k-1}{k}$
$\dim \mathrm{Sym}(X_1, ..., X_n) = \infty$

For completeness: A formal definition.

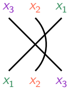
The exterior algebra $\mathrm{Sym}(V)$ of a vector space V over a field (say not of characteristic 2) is defined as the quotient algebra of the tensor algebra $\mathrm{T}(V)$ by the two-sided ideal I generated by the relation

$$X \otimes Y = Y \otimes X$$

- Very often one writes *e.g.* $X \cdot Y$ or XY for the image of $X \otimes Y$ under the canonical surjection $\mathrm{T}(V) \twoheadrightarrow \mathrm{Sym}(V)$
- Note that $X \cdot X \neq 0$ in $\mathrm{Sym}(V)$
- If we choose a basis $\{X_i\}$ of V, then $\mathrm{Sym}(V)$ is the polynomial ring in commuting variables $\{X_i\}$

Some combinatorics in this story

Take the symmetric group and act on $\mathbb{R}[X_1, X_2, X_3]$ by permuting the variables.
Symmetric polynomials? These are fixed by permutation, *e.g.*



$$X_1X_2 + X_1X_3 + X_2X_3 \longrightarrow X_3X_2 + X_3X_1 + X_2X_1 = X_1X_2 + X_1X_3 + X_3X_2$$

degree	polynomials	symmetric basis
0	1	1
1	$X_1 + X_2 + X_3$	$X_1,\ X_2,\ X_3$
2	$X_1X_2 + X_1X_3 + X_2X_3,\ X_1^2 + X_2^2 + X_3^2$	$X_1^2, X_2^2, X_3^2, X_1X_2, X_1X_3, X_2X_3$

These are symmetric polynomials are also called symmetric tensors and they live inside the symmetric algebra (but honestly, so they are not spanning it).

Thank you for your attention!

I hope that was of some help.