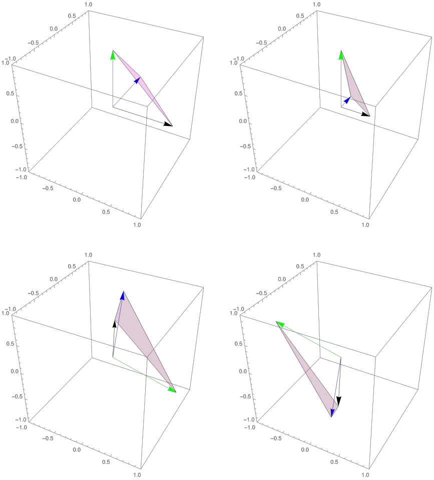
What is...a linear map?

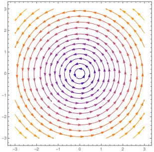
Or: Linear algebra done right.

What is the standard coordinate system?



None of these is better than the others!

Abstract vs. real life

	Abstract	Incarnation
Numbers	3	 or  or...
Linear maps	$f:\mathbb{R}^2\to\mathbb{R}^2, f(x,y)=(y,-x)$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ or  or...

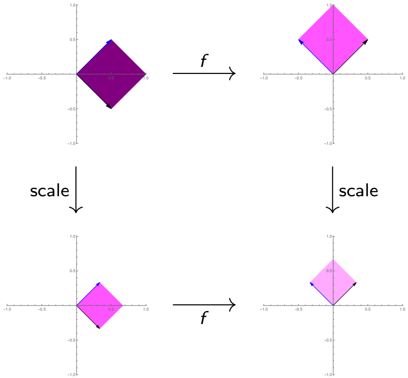
Linear maps are matrices without choosing coordinates

Let us have a look at an example.

$$f : \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) = (y, -x)$$

- f preserves addition, *i.e.*
  $f\big((x, y) + (x', y')\big) = f(x + x', y + y') = (y + y', -x - x') = f(x, y) + f(x', y')$
- f preserves scalar multiplication, *i.e.*
  $f\big(a(x, y)\big) = f(ax, ay) = (ay, -ax) = af(x, y)$

First scale and then rotate is the same as first rotate and then scale:



For completeness: A formal definition.

Let V, W be two vector spaces over $\mathbb{K}$. A map $f\colon V \to W$ is linear if:

- It preserves addition, *i.e.* $f(v+w) = f(v) + f(w)$ for $v, w \in V$
- It preserves scalar multiplication, *i.e.* $f(aw) = af(v)$ for $a \in \mathbb{K}$, $v \in V$

Note that addition of vectors and scalar multiplication are precisely the two operation on vector spaces

In other words, linear maps preserve the vector spaces structures

Choosing bases B_V and B_W of V and W, a linear map f is a matrix $A(f)$ obtained by writing the images of B_V expressed in B_W into the columns of $A(f)$

Everything exists basis free!?

Question. What is a basis free definition of being diagonalizable? Roughly, a linear map $f\colon V \to V$ is called diagonalizable if:

- If there exist scalars $a_i \in \mathbb{K}$ such that

$$(f - a_1)\dots(f - a_n) = 0$$

- There exists a complete orthogonal idempotent decomposition $e_i \in \mathrm{End}_{\mathbb{K}}(V)$ such that

$$(f - a_i)e_i = 0 = e_i(f - a_i)$$

Example. $f(x, y) = (y, x)$, a matrix for f would be $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $a_1 = 1$, $a_2 = -1$

- $(f - 1)(x, y) = (y - x, x - y)$, $(f - \text{-1})(x, y) = (y + x, x + y)$, so

$$(f - 1)(f - \text{-1})(x, y) = (f - 1)(x + y, x + y) = (0, 0)$$

- $e_1(x, y) = (x + y, x + y)$, $e_2(x, y) = (x - y, y - x)$

$$(f - 1)e_1 = 0 = e_1(f - 1), \quad (f - \text{-1})e_2 = 0 = e_2(f - \text{-1})$$

Thank you for your attention!

I hope that was of some help.