What are...direct sums $\oplus$?

Or: How to add vector spaces and matrices.

My wish list for adding vector spaces.

- I want $V \oplus W \cong W \oplus V$.
- I want $(V \oplus W) \oplus X \cong V \oplus (W \oplus X)$.
- I want $\dim(V \oplus W) = \dim(V) + \dim(W)$.

Does this remind you of numbers?

How can we add vectors externally?

$$\mathbb{R}^3 \oplus \mathbb{R}^2 \ni \begin{pmatrix}1\\2\\3\end{pmatrix} \oplus \begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}1\\2\\3\\\hline 4\\5\end{pmatrix} \quad \leftrightsquigarrow \quad \begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix} \in \mathbb{R}^5$$

Vector spaces V and W have bases $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_m\}$.
The space $V \oplus W$ has bases $\{(v_1, 0), ..., (v_n, 0), (0, w_1), ..., (0, w_m)\}$.

Two bases give add to a new one:

$$\left\{ \begin{pmatrix}1\\0\\0\\\hline 0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\hline 0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\\hline 0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\hline 1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\hline 0\\1\end{pmatrix} \right\} \quad \leftrightsquigarrow \quad \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

Where my wishes granted?

$V \oplus W \cong W \oplus V$? Yep:

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \oplus \begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}1\\2\\3\\\hline 4\\5\end{pmatrix} \text{“}=\text{”} \begin{pmatrix}4\\5\\\hline 1\\2\\3\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix} \oplus \begin{pmatrix}1\\2\\3\end{pmatrix}$$

$(V \oplus W) \oplus X \cong V \oplus (W \oplus X)$? Yep:

$$\left(\begin{pmatrix}1\\2\\3\end{pmatrix} \oplus \begin{pmatrix}4\\5\end{pmatrix}\right) \oplus (6) = \begin{pmatrix}\begin{pmatrix}1\\2\\3\\\hline 4\\5\end{pmatrix}\\\hline 6\end{pmatrix} \text{“}=\text{”} \begin{pmatrix}1\\2\\3\\\hline \begin{pmatrix}4\\5\\\hline 6\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \oplus \left(\begin{pmatrix}4\\5\end{pmatrix} \oplus (6)\right)$$

$\dim(V \oplus W) = \dim(V) + \dim(W)$? Yep:

$$\#\left\{\begin{pmatrix}1\\0\\0\\\hline 0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\hline 0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\\hline 0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\hline 1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\hline 0\\1\end{pmatrix}\right\} = 5 = 3 + 2 = \#\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} + \#\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

For completeness: A formal definition.

If V and W are vector spaces, then $V \oplus W$ is the vector space whose:

elements are pairs (v, w) with $v \in V$ and $w \in W$;

addition is componentwise, *i.e.* $(v, w) + (v', w') = (v + v', w + w')$;

scalar multiplication is componentwise, *i.e.* $\lambda(v, w) = (\lambda v, \lambda w)$.

And what about matrices?

$$\left(\mathbb{R}^2 \xrightarrow{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \mathbb{R}^2 \right) \oplus \left(\mathbb{R}^3 \xrightarrow{\begin{pmatrix} 5 & 6 & 7 \end{pmatrix}} \mathbb{R}^1 \right)$$

$$= \mathbb{R}^5 \xrightarrow{\left(\begin{array}{cc|ccc} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 5 & 6 & 7 \end{array}\right)} \mathbb{R}^3$$

They really do not know each other ;-)

Thank you for your attention!

I hope that was of some help.