What is...condensed mathematics?

Or: How can algebra remember topology?

The problem: topology makes algebra harder

Chinese reminder theorem gives:

$\mathbb{Z}/1\mathbb{Z}$	1
$\mathbb{Z}/2\mathbb{Z}$	1
$\mathbb{Z}/3\mathbb{Z}$	1
$\mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2
$\mathbb{Z}/5\mathbb{Z}$	1
$\mathbb{Z}/6\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, 1 \mapsto (1,1)$	1
$\mathbb{Z}/7\mathbb{Z}$	1
$\mathbb{Z}/8\mathbb{Z} \ncong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	3

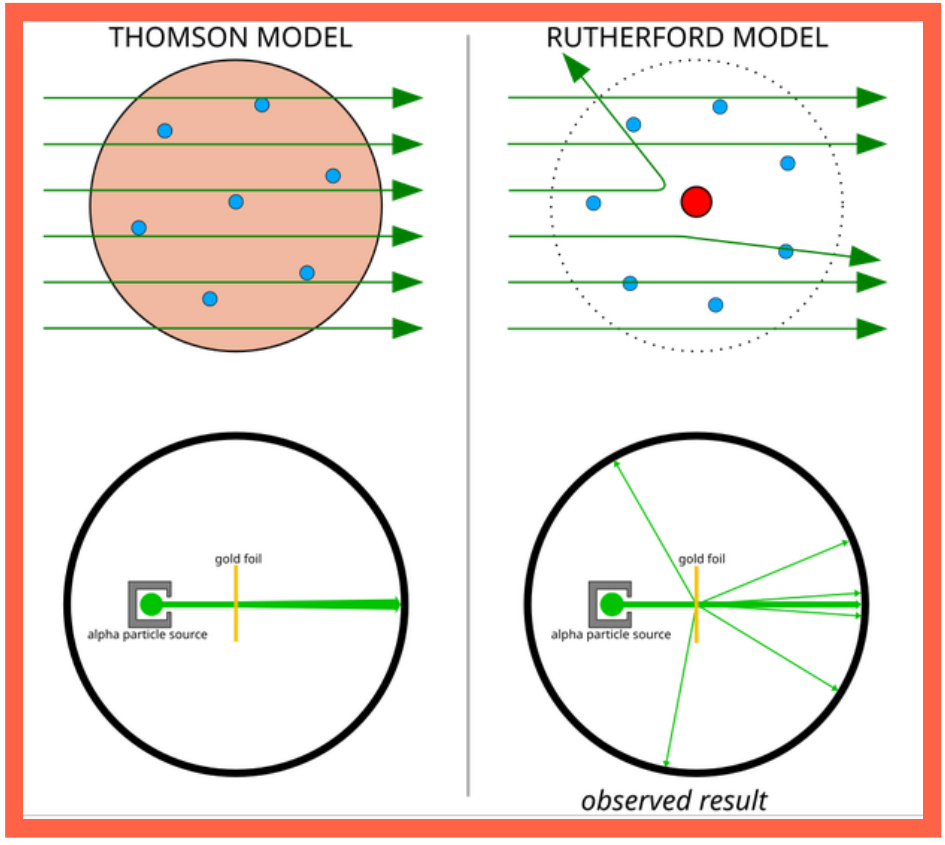
- Example $\mathbb{R}_{\text{discrete}} \longrightarrow \mathbb{R}_{\text{usual}}$ is continuous and bijective
- But The inverse map is not continuous
- Algebra misses this Its kernel and cokernel are both zero

A simple test sees the topology

COVID-19 Antigen Rapid Test Kit;
the timer is provided by the user.

- Test space $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\}$ has $1/n \to 0$
- Usual $\mathbb{R}$ The map $1/n \mapsto 1/n$ is continuous
- Discrete $\mathbb{R}$ A convergent sequence must eventually be constant

2019: condensed sets



- The idea For every profinite S, record $\underline{X}(S) = \mathrm{Cont}(S, X)$
- Gluing Maps that agree on a finite cover glue to one map on S
- Condensed set The collection of all these tests is the new object

Condensed abelian groups behave better

Every map wants a middle

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example of a rectangular matrix in row echelon form

- An epic–monic factorization writes $X \twoheadrightarrow I \hookrightarrow Y$
- For matrices, row reduction finds I; over a field its dimension is the rank
- An abelian category is additive and every morphism has such a factorization

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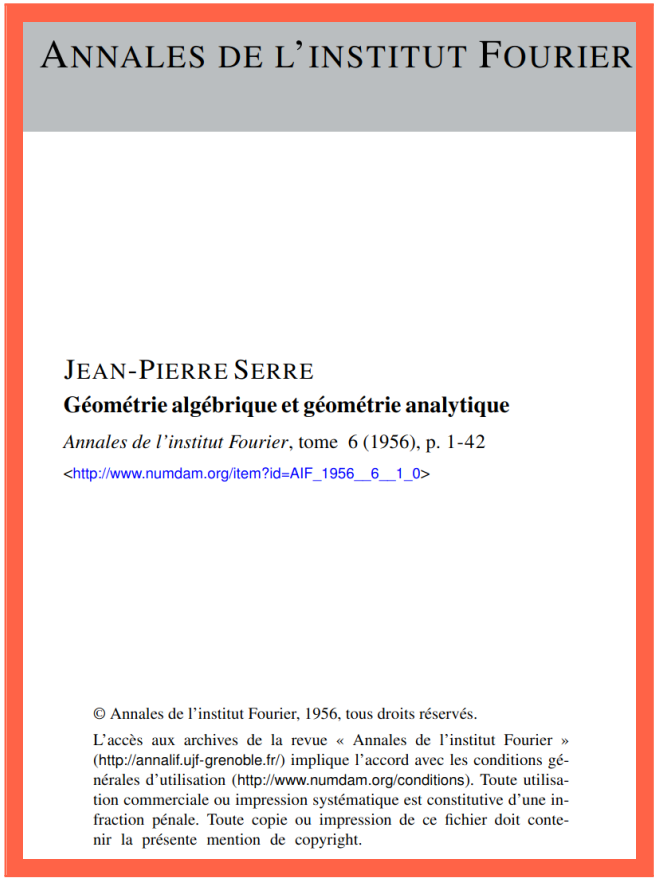
- Our example $0 \to \mathbb{R}_{\text{discrete}} \to \mathbb{R}_{\text{usual}} \to Q \to 0$ now has $Q \neq 0$
- What Q sees $Q(S) = C(S, \mathbb{R})/C_{\text{locally constant}}(S, \mathbb{R})$
- Result Condensed abelian groups form an abelian category; topological abelian groups do not

Applications



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JEAN-PIERRE SERRE

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- Complex geometry GAGA: algebraic and analytic coherent cohomology agree
- p-adic geometry Poincaré duality for $\mathbb{F}_p$-cohomology of rigid spaces
- Abelian groups Condensed $\mathrm{Ext}^1 = 0$ forces A free in ZFC; classically independent of ZFC

Thank you for your attention!

Next time: Perfectoid spaces